q-Series, their Modularity, and Nahm's Conjecture

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Summary

Motivated by the famous Rogers-Ramanujan identities, Nahm asked which generalisations of the Rogers-Ramanujan functions are modular. We define a *Nahm sum* for $A \in \mathbb{Q}^{r\times r}$ positive definite and symmetric, $b \in \mathbb{Q}^r$, and $c \in \mathbb{Q}$ by

$$f_{A,b,c}(q) \;=\; \sum_{n\in\mathbb{Z}^r_{\geq 0}} \frac{q^{\frac{1}{2}n^TAn+b^Tn+c}}{(q)_{n_1}\cdots(q)_{n_r}},$$

where $(q)_n = \prod_{i=1}^n 1-q^i$ is the q-Pochhammer symbol. Nahm sums appear, for example, as characters of conformal field theory, as knot invariants, and as the generating function of classes of partitions.

Based on asymptotic computations, Nahm conjectured that the modularity of Nahm sums is related to the vanishing of the images of the solutions of

$$1-Q_i = \prod_{j=1}^r Q_j^{A_{ij}}, \qquad \text{for } i=1,\ldots,r$$

in the Bloch group. A precise formulation by Zagier turned out to be wrong according to counterexamples given by Vlasenko-Zwegers. We will present generalised asymptotics of Nahm sums $f_{A,b,c}(e^{-h})$ as $h \to 0$ on a ray on the right half-plane. Based on these asymptotics, we will discuss the examples given by Vlasenko-Zwegers and explain how they can be explained in the context of Nahm's observation. Moreover, we will refine the correspondence between the modularity of Nahm sums and the Bloch group by studying the vector-valued modularity of Nahm sums under $\mathrm{SL}_2(\mathbb{Z})$. More precisely, we will describe which $b \in \mathbb{Q}^r$ and $c \in \mathbb{Q}$ have to be chosen such that $f_{A,b,c}(q)$ is modular.

We also discuss applications of generalised Nahm sums.

The tail of the coloured Jones polynomial $\Phi_K(q)$ for an alternating knot K is a q-series knot invariant that arises as a limit of the coloured Jones polynomials. For several knots, $\Phi_K(q)$ is known to be a product of (partial) theta functions and thus "modular". The main result of part II of this thesis is a general formula for $\Phi_K(q)$ in terms of (partial) theta functions for a class of knots.

Part III deals with an application of q-series to the theory of partitions. There, we prove a conjecture of Andrews concerning the sign pattern of coefficients of a q-series from Ramanujan's "lost" notebook. This part is based on a preprint that is joint work with Amanda Folsom, Joshua Males, and Larry Rolen.

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Contents

1. Introduction

In order to study abstract mathematical objects, invariants of the object are important. For example, the character of a vertex operator algebra (VOA) is a q-series invariant and the coloured Jones polynomial of a knot is a polynomial invariant. Those invariants contain information about the abstract object and can be useful to distinguish different objects. The q-series invariants associated to "nice" objects often have modular behaviour, meaning they are either ordinary modular forms or have mock, quasi, or quantum modular behaviour.

I will illustrate the interactions between different objects and q-series using the Rogers-Ramanujan functions

$$G(q) \;=\; \sum_{n\geq 0} \frac{q^{n^2}}{(q)_n}, \qquad H(q) \;=\; \sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} \tag{1.1}$$

where $(q)_n = \prod_{i=1}^{n}(1-q^i)$ is the q-Pochhammer symbol. They satisfy the Rogers-Ramanujan identities

$$G(q) \;=\; \prod_{n=\pm 1 \pmod 5} (1-q^n)^{-1}, \qquad H(q) \;=\; \prod_{n=\pm 2 \pmod 5} (1-q^n)^{-1} \tag{1.2}$$

which were discovered independently by Rogers and Ramanujan and are sometimes considered the "most beautiful pair of formulas in all of mathematics" [Zag19]. The Rogers-Ramanujan functions $G(q)$ and $H(q)$ and their identities appear in numerous different fields of mathematics. Examples include partitions, VOAs, and knot theory:

1. Partitions: The series $G(q), H(q)$ can be interpreted as generating functions for certain partitions. The identity for $G(q)$ implies that the number of partitions such that the parts differ by at least 2 is equal to the number of partitions with parts congruent to ± 1 mod 5. A similar interpretation is true for $H(q)$ and the second identity.

2. VOAs: The graded dimension of one of the easiest rational VOAs, the $(2,5)$–minimal model, is given by $q^{\frac{11}{60}}H(q)$ and the graded dimension for its unique module given is by $q^{-\frac{1}{60}}G(q)$. By counting the graded dimensions in two different ways, Lepowsky-Wilson [LW82] obtained the identities (1.2).

3. Knots: It is known [AD11] that the coloured Jones polynomials for some knots K stabilise to a q-series, $\Phi_K(q)$, the *tail of the coloured Jones polynomial.* For the knot $K = 5_1$, Morton [Mor95] showed that $\Phi_K(q)$ is equal to the product-expansion of $H(q)$, while Armond-Dasbach's work [AD11] implies that $\Phi_K(q)$ is equal to the sum-expansion of $H(q)$. This establishes the second Roger-Ramanujan identity for $H(q)$ using knot theory. We discuss more results concerning $\Phi_K(q)$ for general knots K in Part II of this thesis.

An important consequence of the Rogers-Ramanujan identities in combination with the Jacobi triple product formula is that the functions $q^{-1/60}G(q)$ and $q^{11/60}H(q)$ are modular functions in τ, where $q = e^{2\pi i\tau}$. In two of the mentioned examples, the modularity of (1.1) has a reason within the object: By Zhu's modularity theorem, the graded dimension of a rational VOA are modular functions and so are $q^{-\frac{1}{60}}G(q)$ and $q^{\frac{11}{60}}H(q)$ as the graded dimension of the rational (2, 5)–minimal model. In the context of knot theory, the fact that $K = 5_1$ is a 2–bridge knot ensures that $\Phi_K(q) = H(q)$ is, up to rational powers of q, a theta function and thus modular (cf. Theorem 9.1.3).

This thesis concerns generalisations of the Rogers-Ramanujan functions, their modularity properties, and applications. Part I concerns a family of q-series generalising the Rogers-Ramanujan functions, so-called *Nahm sums* and their modularity properties. In Part II, we consider q-series associated to knots and study their modularity. In Part III, we present some applications of q-series to the theory of partitions.

1.1. Nahm sums

In the context of conformal field theory, Nahm [Nah07] noticed that generalisations of the Rogers-Ramanujan functions appear and studied their modularity properties. For a symmetric, positive definite matrix $A \in \mathbb{Z}^{r\times r}$, $b \in \frac{1}{2}\mathbb{Z}^r$ and $c \in \mathbb{Q}$, the *Nahm sum* $f_{A,b,c}(q)$ is defined by

$$f_{A,b,c}(q) \;=\; \sum_{n\in\mathbb{Z}^r_{\geq 0}} \frac{q^{\frac{1}{2}n^TAn+b^Tn+c}}{(q)_{n_1}\cdots(q)_{n_r}} \in q^c\mathbb{Z}[[q^{\frac{1}{2}}]] \tag{1.3}$$

where $q = e^{2\pi i\tau}$ and $\tau \in \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Generalisations of Nahm sums are also of great interest, cf. Sections 3.4 and 3.4, but are omitted here for the sake of simplicity.

It turns out that the modularity of Nahm sums is closely related to algebraic K-theory, more precisely to the vanishing of certain elements in the Bloch group. In this thesis, we will present our contributions to a better understanding of this relationship.

1.1.1. Modularity of Nahm sums and the Bloch group

The Rogers-Ramanujan identities, as well as several examples from conformal field theory, motivated Nahm to ask for which triples (A, b, c) the function $f_{A,b,c}$ is modular for some congruence subgroup. In [Nah07], he observes that if $Q = (Q_1, \dots, Q_r) \in \mathbb{C}^r$ is a solution of

$$1 - Q_i \;=\; \prod_{j=1}^{r} Q_j^{A_{ij}}, \qquad i = 1, \dots, r, \tag{1.4}$$

then $[Q] = \sum_{i=1}^{r}[Q_i] \in \mathbb{Z}[\mathbb{C}]$ is an element in the Bloch group

$$\mathcal{B}(\mathbb{C}) \;\subset\; \mathbb{Z}[\mathbb{C}]/([X] + [1 - X], \dots). \tag{1.5}$$

For a fixed A as above, we write $\mathcal{Q}_A$ for the set of solutions of (1.4). Based on asymptotic computations and several examples coming from conformal field theory, Nahm [Nah07] made the following remarkable but unprecise prediction: The modularity of (1.3) is connected to the vanishing of the images of *some* solutions $Q \in \mathcal{Q}_A$ of (1.4) in the Bloch group $\mathcal{B}(\mathbb{C})$. A precise theorem for $r = 1$ was proved by Zagier in [Zag07] by classifying all modular Nahm sums.

Theorem 1.1.1 (Zagier, 2006)**.** *The function $f_{A,b,c}$ for $(A, b, c) \in \mathbb{Q}_{>0}\times\mathbb{Q}\times\mathbb{Q}$ is modular for some congruence subgroup exactly in* 7 *cases and only for $A \in \{\frac{1}{2}, 1, 2\}$.*

As one can show, the images of the solutions of $1 - Q = Q^A$ vanish in $\mathcal{B}(\mathbb{C})$ if and only if $A \in \{\frac{1}{2}, 1, 2\}$. Hence, Zagier's Theorem proves a rigorous version of Nahm's prediction for $r = 1$. To motivate Nahm's observation, we consider the asymptotics of $f_{A,b,c}(\tau)$ as $\tau \to 0$ on the upper half-plane. Different versions of the following result can be found in [GZ21, Mei54, VZ11, Zag07].

Theorem 1.1.2. *Let A, b, c be as above. Then as $h \searrow 0$,*

$$f_{A,b,c}(e^{-h}) \;=\; e^{V(Q^{(1)})/h}\,\Phi^{(Q^{(1)})}_{A,b,c}(h)\,(1 + O(h^L)) \tag{1.6}$$

for all $L > 0$, where $Q^{(1)} \in \mathcal{Q}_A$ is the unique solution of (1.4) in $(0,1)^r$, $\Phi^{(Q^{(1)})}_{A,b,c}(h) \in \overline{\mathbb{Q}}[[h]]$ is a power series with algebraic coefficients, and $V(Q^{(1)}) \in \mathbb{R}$ is essentially the linear extension of the Rogers-dilogarithm of $Q^{(1)}$.

If $f(q)$ is modular in τ of weight $k \in \mathbb{Z}$ where $q = e^{2\pi i\tau}$, then the asymptotics as $h \to 0$ has the special form

$$f(e^{-h}) \;=\; a_0 h^{-k} e^{\alpha/h}(1 + o(h^L)), \tag{1.7}$$

for all $L > 0$ and some $a_0 \in \mathbb{C}$, $\alpha \in 4\pi^2\mathbb{Q}$. Hence, if $f = f_{A,b,c}$, we conclude that $k = 0$, the power series $\Phi^{(Q^{(1)})}_{A,b,c}(h)$ is constant, and $V(Q^{(1)}) \in 4\pi^2\mathbb{Q}$. The last condition is expected to be equivalent to the vanishing of $[Q^{(1)}]$ in $\mathcal{B}(\mathbb{C})$. This suggests that the following statement, which was part of Nahm's observations and was recently proved by Calegari, Garoufalidis, and Zagier, is true.

Theorem 1.1.3 ([CGZ23])**.** *If $f_{A,b,c}$ is modular, then $[Q^{(1)}]$ vanishes in $\mathcal{B}(\mathbb{C})$.*

It is known from [Zag07] that the vanishing of $[Q^{(1)}]$ in $\mathcal{B}(\mathbb{C})$ is not sufficient for the modularity. Hence, a reasonable guess would be that for a fixed matrix A, the modularity of $f_{A,b,c}$ for some b, c is equivalent to the vanishing of *all* solutions of (1.4) in $\mathcal{B}(\mathbb{C})$. This was conjectured in [Zag07], but Vlasenko and Zwegers gave counterexamples in [VZ11]. We discuss one of these examples in more detail below.

1.1.2. Asymptotics of Nahm sums on rays in the upper half-plane

Similar to the contribution for $Q^{(1)}$ in Theorem 3.2.5, we can compute the formal asymptotic contributions corresponding to all solutions $Q \in \mathcal{Q}_A$ of (1.4). In Chapter 4, we associate to any $Q \in \mathcal{Q}_A$ a *volume* $V(Q) \in \mathbb{C}$, essentially given by a continuation of the Rogers-dilogarithm, and a power series $\Phi^{(Q)}_{A,b}(h) \in \overline{\mathbb{Q}}[[h]]$.

Instead of considering the radial asymptotics $h \searrow 0$ as in Theorem 1.1.2, it turns out to be helpful to consider the asymptotics of $f_{A,b}(e^{-h})$ as $h \to 0$ on rays in the right half-plane. Then some of the power series $e^{V(Q)/h}\Phi^{(Q)}_{A,b}(h)$ associated to $Q \in \mathcal{Q}_A$ occur in the asymptotic expansion.

Theorem (Theorem 4.3.1)**.** *As $h \to 0$ on a ray in the right half-plane, we have*

$$f_{A,b}(e^{-h}) \;=\; \sum_{Q \in \mathcal{Q}_A} e^{V(Q)/h}\,\Phi^{(Q)}_{A,b}(h)\;(1 + O(h^L)). \tag{1.8}$$

for all $L > 0$.

Here, the sum in the asymptotic expansion should be interpreted as a sum of formal asymptotic contributions and only the leading terms with $\mathrm{Re}(V(Q)/h)$ maximised are relevant for the asymptotics. In Section 5.2, we will consider an example where the subleading contributions are numerically visible. This is related to the quantum modularity of Nahm sums studied by Wheeler in his thesis [Whe23].

1.1.3. Vector-valued modularity

Even though Theorem 1.1.1 seems to solve the problem for $r = 1$, it is not completely enlightening for which b's and c's $f_{A,b,c}$ is modular. Furthermore, it only claims modularity for *some* congruence subgroup. It turns out that it is helpful to consider $f_{A,b,c}$ as a component of a vector-valued modular function for $SL_2(\mathbb{Z})$. For instance, for the Rogers-Ramanujan functions (1.1) it is well known that the vector-valued function $F_2(\tau) = (q^{-\frac{1}{60}}G(q),\, q^{\frac{11}{60}}H(q))^t$, where $q = e^{2\pi i\tau}$, is a modular function for the full modular group $\mathrm{SL}_2(\mathbb{Z})$:

$$F_2(\tau+1) \;=\; \begin{pmatrix} \zeta_{60}^{-1} & 0 \\ 0 & \zeta_{60}^{11} \end{pmatrix} F_2(\tau), \qquad F_2\left(-\frac{1}{\tau}\right) \;=\; \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\frac{2\pi}{5} & \sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & -\sin\frac{2\pi}{5} \end{pmatrix} F_2(\tau), \quad (1.9)$$

where $\zeta_{60} = e^{2\pi i/60}$.

For simplicity, we assume in the following that A is an even matrix and $b \in \mathbb{Z}^r$. If we compute hidden terms, either numerically or formally, in the asymptotic expansion for a general Nahm sum $f_{A,b}(e^{-h})$ as $h \to 0$ in (1.8), we see contributions from all solutions $Q \in \mathcal{Q}_A$. Moreover, for every solution, there is an attached $\tilde{q}$-series, where $\tilde{q} = e^{-4\pi^2/h}$,

$$f_{A,b,c}(e^{-h}) \;\sim\; \sum_{Q\in\mathcal{Q}_A} \Phi_{A,b,c}^{(Q)}(h)\; \tilde{q}^{c(Q)}(1 + a_{Q,1}\tilde{q} + a_{Q,2}\tilde{q}^2 + \ldots)$$

where $c(Q) = -V(Q)/4\pi^2$. It turns out that the $\tilde{q}$-series have the form $f_{A,b(Q),c(Q)}(\tilde{q})$ for some known $b(Q) \in \mathbb{Z}^r$.

If $f_{A,b,c}(q)$ is modular (of weight 0), then $f_{A,b,c}(\tilde{q})$ is a power series in fractional powers of $\tilde{q}$, so that all power series $\Phi_{A,b,c}^{(Q)}(h)$ that contribute in the expansion should be constant. This indicates that the vector-valued function

$$F_A(q) \;=\; \left(f_{A,b(Q),c(Q)}(q)\right)_{Q\in\mathcal{Q}_A} \tag{1.10}$$

is a vector-valued modular function for $\mathrm{SL}_2(\mathbb{Z})$. The transformation of F_A under $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ is easy (the Q-component is simply multiplied by $e^{2\pi i c(Q)}$), and the transformation under $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ can be given in terms of the solutions $Q \in \mathcal{Q}_A$ of equation (1.4).

We illustrate the observation in the following example.

Example 1.1.4. We consider again the Rogers-Ramanujan functions, i.e., $A = 2$ and F_2 defined as above. The equation (1.4) has the two solutions $Q^{(1)} = \frac{-1+\sqrt{5}}{2}$ and $Q^{(2)} = \frac{-1-\sqrt{5}}{2}$ with Rogers-dilogarithms $\frac{1}{60}4\pi^2$ and $-\frac{11}{60}4\pi^2$. They coincide with the values for $-4\pi^4 c \in \pi^2\mathbb{Q}$ such that $f_{2,b,c}$ is modular. Moreover, we can compute the matrix in (1.9) using

$$\begin{aligned} \frac{2}{\sqrt{5}}\sin\left(\frac{2\pi}{5}\right) \;&=\; \frac{1}{\sqrt{2-Q^{(1)}}} \;=\; \frac{-Q^{(2)}}{\sqrt{2-Q^{(2)}}}, \\ \frac{2}{\sqrt{5}}\sin\left(\frac{\pi}{5}\right) \;&=\; \frac{1}{\sqrt{2-Q^{(2)}}} \;=\; \frac{Q^{(1)}}{\sqrt{2-Q^{(1)}}}. \end{aligned} \tag{1.11}$$

We discuss more details concerning the vecor-valued modularity in Chapter 5.

1.1.4. Nahm's conjecture revisited

We consider the four-dimensional Nahm sum for

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad b = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \qquad c = \frac{1}{15}$$

from [VZ11]. It is known that $f_{A,b,c}$ is modular. Equation (1.4) for A has in total eight solutions, four of the form

$$\left(u, u, \frac{1}{1+u}, \frac{1}{1+u}\right), \quad \text{with } 1-u^2 = u^4 \tag{1.12}$$

whose images vanish in $\mathcal{B}(\mathbb{C})$ and four solutions of the form

$$\left(u, -u, \frac{1}{1+u}, \frac{1}{1-u}\right), \quad \text{with } 1-u^2 = -u^4, \tag{1.13}$$

i.e., u is a root of unity of order 12, whose images do not vanish in $\mathcal{B}(\mathbb{C})$. In view of Theorem 4.3.1, we can associate a completed power series $e^{V(Q)/h}\Phi^{(Q)}_{A,b}(h)$ to the above solutions. For the solutions Q as in (1.13) corresponding to $u = \pm e^{\pi i/6}$ (resp. $u = \pm e^{\pi i 5/6}$), we find that they have an asymptotic contribution $\pm e^{V(Q)/h}\Phi^{(Q)}_{A,b}(h)$ and thus they cancel each other in the asymptotic expansion. Hence, the contributions corresponding to all solutions with non-vanishing class in $\mathcal{B}(\mathbb{C})$ are not relevant for the modularity. We discuss this example in more detail in Section 6.2.

Using the refined asymptotics of Nahm sums for $\tau \to 0$ on a fixed ray in the upper half-plane, we can prove a conditional result.

Theorem (Theorem 4.3.4). *Assume that $V(Q) \neq V(Q')$ for all $Q, Q' \in \mathcal{Q}_A$. If $f_{A,b,c}$ is modular, then $[Q] = 0$ for all $Q \in \mathcal{Q}_A$.*

The idea of the proof is that if $f_{A,b,c}(q)$ is modular, then $\mathrm{Im}(V(Q)) = 0$ for all $Q \in \mathcal{Q}_A$. Because $\mathrm{Im}(V(Q))$ is the Bloch-Wigner dilogarithm of Q, this implies that $[Q] = 0$ for all $Q \in \mathcal{Q}_A$ (see Theorem 2.2.3).

1.1.5. Modular combinations of Nahm sums

If for a given matrix A the Nahm sums $f_{A,b,c}$ are not modular for any $b \in \mathbb{Q}^r, c \in \mathbb{Q}$, it is still possible that a linear combination of $f_{A,b,c}$'s is modular. In agreement with Nahm's observation, this can happen if $[Q] = 0$ for some but not necessarily all $Q \in \mathcal{Q}_A$.

In Section 6.3, we provide new examples of this phenomenon and discuss them in the context of Nahm's conjecture. For instance, for $A = \left(\begin{smallmatrix} 8 & 5 \\ 5 & 4 \end{smallmatrix}\right)$ the solutions $\mathcal{Q}_A$ have both vanishing and non-vanishing images in $\mathcal{B}(\mathbb{C})$ and $f_{A,b,c}$ is never modular. However, the linear combination

$$q^{-1/60}\left(f_{\left(\begin{smallmatrix} 8 & 5 \\ 5 & 4 \end{smallmatrix}\right),\left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}\right)}(q) \; - f_{\left(\begin{smallmatrix} 8 & 5 \\ 5 & 4 \end{smallmatrix}\right),\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)}(q) \; + f_{\left(\begin{smallmatrix} 8 & 5 \\ 5 & 4 \end{smallmatrix}\right),\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)}(q)\right) \;=\; q^{-1/60}G(q) \tag{1.14}$$

with $q = e^{2\pi i\tau}$ is a modular function in τ. We end with a possible version of Nahm's conjecture in Section 6.3: For a given A as above, the vanishing of the image of some solutions $Q \in \mathcal{Q}_A$ in $\mathcal{B}(\mathbb{C})$ implies the modularity of a linear combination of Nahm sums.

1.2. Knots and q-series

Part II of this thesis concerns q-series that appear in the context of knot theory and their modularity properties.

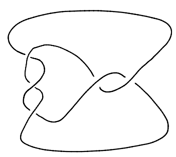
The coloured Jones polynomials of a knot are important quantum knot invariant. For instance, the Kashaev invariant can be defined in terms of the coloured Jones polynomial and according to the quantum modularity conjecture, the Kashaev invariant is believed to posses quantum modular behaviour [GZ23, GZ24, Zag10]. The quantum modularity conjecture is in fact related to the asymptotics of Nahm sums discussed in Part I.

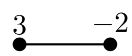
In addition, properties of a limit of the coloured Jones polynomial for some knots have been studied. For alternating knots, it is known [AD11] that the coloured Jones polynomials stabilise, after multiplying by suitable powers of q, to a q-series $\Phi_K(q)$, called the *tail of the coloured Jones polynomial*, so that the first N coefficients of the N-th coloured Jones polynomial agree with the first N coefficients of $\Phi_K(q)$. In this case, the q-series $\Phi_K(q)$ can be written as an explicit q-hypergeometric series that can be seen as a generalisation of Nahm sums [GL15].

In [GL15, Appendix D], Garoufalidis and Lê with Zagier recognised that the tails of the coloured Jones polynomial for almost all knots with up to 8 crossings can be written as products of the functions $h_b(q) = \sum_{n\in\mathbb{Z}} \epsilon_b(n) q^{bn(n+1)/2-n}$, $\epsilon_b(n) \in \{\pm 1\}$. These functions are, up to a rational power of q, theta functions if b is odd and partial theta functions if b is even. These identities have been proven by Andrews [And13] and Keilthy-Osburn [KO16] and generalised to knots with up to 10 crossings by Beirne-Osburn [BO17].

The main result in Part II is Theorem 9.1.3, a general formula for the tail of the coloured Jones Polynomial in terms of $h_b(q)$ for classes of alternating *arborescent* knots. This result, which is the author's work, will also appear in a paper with Osburn [OS24].

Following [BS10], we can associate a knot K to any weighted tree $(\mathcal{V}, \mathcal{E}, w)$, $w : \mathcal{V} \to \mathbb{Z}$, and any knot that is constructed in this way is called *arborescent*. For example, the knot $K = 5_2$ and an associated graph are given by





(1.15)

We assume that Γ has no vertices with degree ≤ 2 of weight 0, which implies that K is a prime knot. If there exists a bipartition $\mathcal{V} = \mathcal{V}_+ \cup \mathcal{V}_-$ with $\pm w(\mathcal{V}_\pm) \geq 0$, then K is alternating and we call the weighted tree alternating.

Theorem (Theorem 9.1.3)**.** *Let $(\mathcal{V}, \mathcal{E}, w)$ be an alternating, weighted tree and K the associated alternating knot. If $0 \notin w(\mathcal{V}_-)$, then*

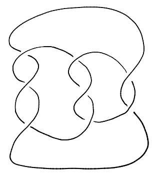
$$\Phi_K(q) \;=\; \prod_{v\in\mathcal{V}_+} h_{w(v)+e(v)}(q) \tag{1.16}$$

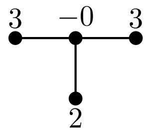
where $e(v)$ denotes the number of edges adjacent to $v \in \mathcal{V}$.

For example, for $K = 5_2$ as in (1.15), Theorem 9.1.3 implies

$$\Phi_K(q) \;=\; h_4(q) \;=\; 1 - q + q^3 - q^6 + q^{10} - q^{15} + O(q^{20}). \tag{1.17}$$

The first example where it is unknown whether the tail of the coloured Jones polynomial has a representation as a product of h_b's is $K = 8_5$. A knot diagram for 8_5 and an associated weighted tree are given by





Because $0 \in w(\mathcal{V}_-)$, Theorem 9.1.3 is not applicable. It is easy to check that (1.16) is not true for $\Phi_{8_5}(q)$ and numerical computations presented in Section 9.4 strongly suggest that $\Phi_{8_5}(q)$ cannot be written in terms of h_b's. This leads to the following problem.

Question 1.2.1. *Does Theorem 9.1.3 classify all modular tails of the coloured Jones polynomials? In other words, for alternating, arborescent knots K, is it true that $\Phi_K(q)$ is a product of (partial) theta functions if and only if there exists an associated weighted tree with $0 \notin w(\mathcal{V}_-)$?*

1.3. Partitions and q-series

Part III of this thesis deals with applications of generalised Nahm sums and more general q-series to the theory of partitions.

As illustrated with the Rogers-Ramanujan functions, one-dimensional Nahm sums are the generating function for certain partitions. By modifying the sum slightly, the coefficients will count the partitions in a slightly different way. I will present an example that deals with a problem posed by Andrews concerning a generating function for partitions that I recently solved in joint work with Folsom, Males, and Rolen [FMRS23].

In 1986, Andrews [And86] considers the coefficients of the function

$$v_1(q) \;=\; \sum_{n\geq 0} \frac{q^{\frac{1}{2}n(n+1)}}{(-q^2;q^2)} \;=:\; \sum_{n\geq 0} V_1(n)q^n \tag{1.18}$$

from Ramanujan's lost notebook [And84]. They count the difference between odd-even partitions (i.e., partitions in which the parity of the parts alternates with the smallest part odd) with rank congruent to 0 mod 4 and 2 mod 4. Andrews notes that they have "a lengthy sign change pattern that alters fairly infrequently" and makes some conjectures. For example, he conjectures that "for almost all n, $V_1(n), V_1(n+1), V_1(n+2)$, and $V_1(n+3)$ are two positive and two negative numbers".

We are able to solve the problem using the asymptotics of generalised Nahm sums like v_1 and an adapted circle method. More precisely, we obtain as $n \to \infty$

$$V_1(n) \;=\; (-1)^{\lfloor \frac{n}{2} \rfloor}(\gamma_1 \pm \gamma_2)\,\frac{e^{c\sqrt{n}}}{\sqrt{n}}\big(\cos(c\sqrt{n}) \pm \sin(c\sqrt{n})\big) + \mathrm{O}\bigg(\frac{e^{c\sqrt{n}}}{n^{3/2}}\bigg), \tag{1.19}$$

where $\pm = (-1)^n$, $c = \sqrt{2\mathrm{Im}(\mathrm{Li}_2(e^{\pi i/3}))/8} = 0.50372\cdots$, and γ_1, γ_2 are explicit real numbers. This explains Andrews' observations and proves some of his conjectures.

Theorem (Theorem 10.2.4)**.** *Two of Andrews's conjectures are true:*

1. *We have $|V_1(n)| \to \infty$ as $n \to \infty$ away from a set of density 0.*
2. *For almost all n, $V_1(n), V_1(n+1), V_1(n+2)$, and $V_1(n+3)$ are two positive and two negative numbers.*

Moreover, using the asymptotics of the coefficients, we can predict when their sign pattern changes.

Part I.

Nahm sums and their modularity

2. Background

2.1. The dilogarithm function

The two main players in this thesis are Nahm sums and the Bloch group. The dilogarithm function will play the role of the intermediary between them. In this section, we will discuss the dilogarithm, as well as the polylogarithms, and discuss their basic properties. Most of the results recalled here are presented in a nice way in [Zag07].

For $m \in \mathbb{Z}$, the *m-th polylogarithm* is definded for $z \in \mathbb{C}$ with $|z| < 1$ via the power series

$$\mathrm{Li}_m(z) \;:=\; \sum_{n=1}^{\infty} \frac{z^n}{n^m}. \tag{2.1}$$

The polylogarithms fulfil the relation

$$\frac{d}{dz}\,\mathrm{Li}_m(z) \;=\; \frac{1}{z}\,\mathrm{Li}_{m-1}(z). \tag{2.2}$$

For $m = 1$, we obtain the logarithm

$$\mathrm{Li}_1(z) \;=\; -\log(1-z) \tag{2.3}$$

and for $m = 2$ the *dilogarithm function*. Since $\log(1-z)$ has an analytic continuation to $\mathbb{C} \smallsetminus [1,\infty)$, we can continue Li_2 analytically as well via

$$\mathrm{Li}_2(z) \;=\; -\int_0^z \frac{1}{t}\log(1-t)dt \quad \text{for } z \in \mathbb{C} \smallsetminus [1,\infty), \tag{2.4}$$

where the contour is given as in Figure 2.1. Since $\log(1-z)$ jumps by $2\pi i$ when z crosses the cut at $[1,\infty)$, $\mathrm{Li}_2(z)$ jumps by $2\pi i \log(z)$ when z crosses the cut at $[1,\infty)$. More generally, we have

$$\mathrm{Li}_m(z) \;=\; \int_0^z \frac{1}{t}\,\mathrm{Li}_{m-1}(t)dt \qquad \text{for } m \in \mathbb{Z}. \tag{2.5}$$

For $m \le 0$, the polylogarithm $\mathrm{Li}_m(z)$ is a rational function in z with poles of order $-m+1$ at $z = 1$. For example,

$$\mathrm{Li}_0(z) \;=\; \frac{z}{1-z}, \qquad \mathrm{Li}_{-1}(z) \;=\; \frac{z}{(1-z)^2}, \qquad \mathrm{Li}_{-2}(z) \;=\; \frac{z(1+z)}{(1-z)^3}. \tag{2.6}$$

We also consider the function

$$F(v) \;=\; \mathrm{Li}_2(1-e^v) \tag{2.7}$$

for $\in \mathbb{C}$. Because its derivative $\frac{v}{e^{-v}-1}$ is meromorphic with poles at $v = 2\pi i n \in 2\pi i\mathbb{Z}$ and residues $-2\pi i n$, the function F is a well defined function modulo $4\pi^2\mathbb{Z}$.

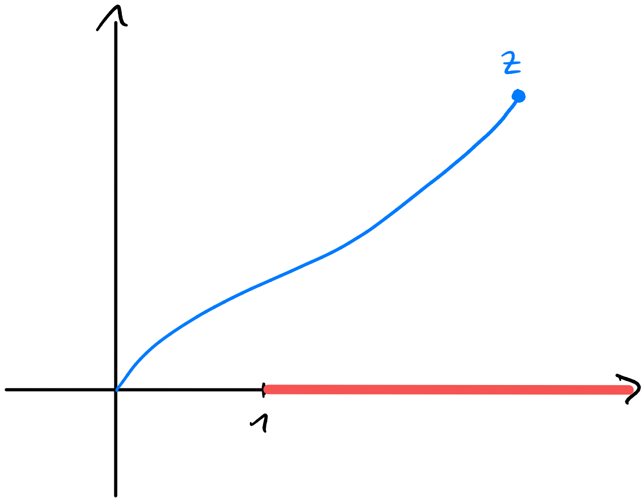


Figure 2.1.: The contour of integration in (2.4) and (2.5)

2.1.1. Functional equations

The dilogarithm function fulfils several functional equations, for example the reflection properties

$$\begin{aligned} \mathrm{Li}_2\left(\frac{1}{z}\right) + \mathrm{Li}_2(z) &= -\frac{\pi^2}{6} - \frac{1}{2}\log^2(-z), \\ \mathrm{Li}_2\left(1-z\right) + \mathrm{Li}_2\left(z\right) &= \frac{\pi^2}{6} - \log(z)\log(1-z) \end{aligned} \tag{2.8}$$

for $z \in \mathbb{C}$ avoiding the branch cuts. Moreover, the dilogarithm function fulfils the 5–term relation for $x, y \in \mathbb{C}$ with $xy \neq 1$

$$\begin{aligned} &\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2\left(\frac{1-x}{1-xy}\right) + \mathrm{Li}_2\left(1-xy\right) + \mathrm{Li}_2\left(\frac{1-y}{1-xy}\right) \\ = &\frac{\pi^2}{6} - \log(x)\log(1-x) - \log(y)\log(1-y) + \log\left(\frac{1-x}{1-xy}\right)\log\left(\frac{1-y}{1-xy}\right). \end{aligned} \tag{2.9}$$

The dilogarithm also fulfils the duplication property

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(-z) = \frac{1}{2}\mathrm{Li}_2\left(z^2\right), \tag{2.10}$$

as well as the distribution property

$$\mathrm{Li}_2(x) = n\sum_{z^n=x}\mathrm{Li}_2(z). \tag{2.11}$$

2.1.2. Variants of the dilogarithm function

We also consider different variants of the dilogarithm function.

The Bloch-Wigner dilogarithm

From the branching of Li_2, it follows that the *Bloch-Wigner dilogarithm*, defined by

$$D(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|, \tag{2.12}$$

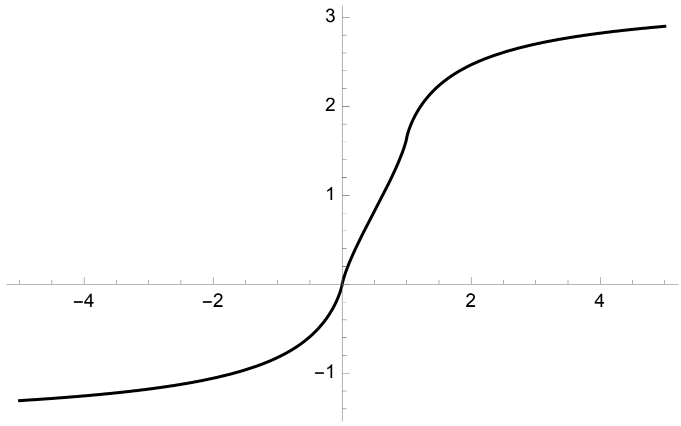


Figure 2.2.: The Rogers dilogarithm $L(x)$ for $x \in (-5, 5)$

is well defined and real analytic on $\mathbb{C} \smallsetminus \{0, 1\}$. Moreover, the Bloch-Wigner dilogarithm satisfies clean versions of the functional equations (2.8) and (2.9), meaning

$$\begin{aligned} D\left(\frac{1}{z}\right) + D(z) &= D(1-z) + D(z) = 0, \\ D(x) + D(y) + D\left(\frac{1-x}{1-xy}\right) + D(1-xy) + D\left(\frac{1-y}{1-xy}\right) &= 0. \end{aligned} \tag{2.13}$$

The Rogers dilogarithm

The *Rogers dilogarithm* is defined for $x \in (0, 1)$ by

$$L(x) := \mathrm{Li}_2(x) + \frac{1}{2}\log(x)\log(1-x) \tag{2.14}$$

and can be extended with $L(0) = 0$ and $L(1) = \frac{\pi^2}{6}$ to all of $\mathbb{R}$ by setting

$$L(x) = \begin{cases} 2L(1) - L\left(\frac{1}{x}\right) & \text{if } x > 1, \\ -L\left(\frac{x}{x-1}\right) & \text{if } x < 0. \end{cases} \tag{2.15}$$

Then L is a continuous function on $\mathbb{R}$ that is analytic on $\mathbb{R} \smallsetminus \{0, 1\}$. The limits as $x \to \pm\infty$ are given by

$$\lim_{x\to\infty} L(x) = \frac{\pi^2}{3}, \qquad \lim_{x\to-\infty} L(x) = -\frac{\pi^2}{6}. \tag{2.16}$$

The function $L(x)$ for $x \in (-5, 5)$ is depicted in Figure 2.2.

2.2. The Bloch group

In this section, we introduce the Bloch group. The Bloch group occurs in the context of algebraic K-theory, hyperbolic geometry, polylogarithms, and, as we will see, modular functions.

For a formal linear combination $[z_1] + \ldots + [z_n]$ with $z_1, \ldots, z_n \in \mathbb{C} \smallsetminus \{0, 1\}$, we define the operation

$$\begin{aligned} d : \mathbb{Z}[\mathbb{C}] &\to \wedge^2(\mathbb{C} \setminus \{0\}) \\ \sum_i [z_i] &\mapsto \sum_i z_i \wedge (1 - z_i)\,. \end{aligned} \tag{2.17}$$

Here, $\wedge^2(\mathbb{C} \setminus \{0\})$ denotes the set of formal linear combinations of the form $x \wedge y$ for $x, y \in \mathbb{C} \setminus \{0\}$, together with the relations $(x \wedge y) + (y \wedge x) = 0$ and $(x_1 x_2) \wedge y = x_1 \wedge y + x_2 \wedge y$. For $(x, y) \in P^1(\mathbb{C})^2 \setminus \{(0, \infty), (\infty, 0), (1, 1)\}$, set

$$s(x, y) \;=\; [x] + [y] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right] + [1 - xy] \;\in\; \mathbb{Z}[\mathbb{C}], \tag{2.18}$$

where we use the convention $[0] = [\infty] = 0$. Then it is easy to see that $s(x, y) \in \ker d$. Let $\mathcal{S}$ be the submodule generated by elements of the form $s(x, y)$ as above. We define the *Bloch group* as the quotient

$$\mathcal{B}(\mathbb{C}) \;=\; \ker(d)/\mathcal{S}. \tag{2.19}$$

There are numerous relations in the Bloch group coming from the 5-term relation (2.18). We illustrate some relations in the next example.

Example 2.2.1. 1. By setting $(x, y) = (1, \infty)$ in (2.18), we obtain $2[1] = 0$.

2. With $(x, y) = (x, 0)$, resp. $(x, y) = (x, \infty)$, for $x \in \mathbb{C}$, we obtain from (2.18)

$$[x] + [1 - x] \;=\; [x] + \left[\frac{1}{x}\right] \;=\; 0. \tag{2.20}$$

We deduce that $2[\frac{1}{2}] = [2] + [\frac{1}{2}] = 0 \in \mathcal{B}(\mathbb{C})$.

3. Let $Q = \frac{-1 \pm \sqrt{5}}{2}$ be a solution of $1 - Q = Q^2$. Then

$$Q \wedge (1 - Q) \;=\; Q \wedge Q^2 \;=\; 2(Q \wedge Q) \;=\; 0 \tag{2.21}$$

and the image $[Q]$ of Q lies in $\mathcal{B}(\mathbb{C})$. Furthermore, we have with $x = y = Q$ in (2.18)

$$s(Q, Q) \;=\; [Q] + [Q] + \left[\frac{1-Q}{1-Q^2}\right] + \left[\frac{1-Q}{1-Q^2}\right] + \left[1 - Q^2\right] \;=\; 5[Q] \tag{2.22}$$

and thus $5[Q] = 0$ in $\mathcal{B}(\mathbb{C})$. We will see below that $[Q]$ even vanishes in $\mathcal{B}(\mathbb{C})$.

4. We also construct a new relation that will appear in Section 3.4. If we apply the 5-term relation (2.18) to the elements $1 - x$ and $1 - y$ for $x, y \in \mathbb{C} \setminus \{0, 1\}$ with $(1 - x)(1 - y) \neq 1$, we obtain

$$0 \;=\; [1 - x] + [1 - y] + \left[\frac{x}{x + y - xy}\right] + [x + y - xy] + \left[\frac{x}{x + y - xy}\right] \tag{2.23}$$

and thus, from (2.20) by adding $[x] + [y]$ on both sides,

$$[x] + [y] \;=\; \left[\frac{x}{x + y - xy}\right] + \left[\frac{y}{x + y - xy}\right] + [x + y - xy]. \tag{2.24}$$

Comparing the elements in equation (2.18) with the functional equations of the Bloch-Wigner dilogarithm in (2.13), we see that D extends to a well-defined linear map on $\mathcal{B}(\mathbb{C})$. For $\xi = \sum_j [Q_j] \in \mathcal{B}(\mathbb{C})$ we write $D(\xi) = \sum_j D(Q_j)$. If $Q_j \in \mathbb{R}$, the 5-term relation (2.9) implies that $L(\xi) = \sum_j L(Q_j)$ is a well-defined real number, where L is the Rogers dilogarithm defined in (2.14).

One can also define the Bloch group $\mathcal{B}(F)$ of a number field F by replacing $\mathbb{C}$ in the construction by F. Then $\mathcal{B}(F)$ agrees, up to torsion, with the algebraic K_3-group of F ([Sus90]). Since this aspect is not relevant to this thesis, we will only consider $\mathcal{B}(\mathbb{C})$ here. A theorem by Merkurjev and Suslin [SM91] states that if for a number field F an element $\xi \in \mathcal{B}(F)$ is torsion, then ξ vanishes in $\mathcal{B}(F')$ for a number field F' containing F and sufficiently many roots of unity. This implies the following result.

Theorem 2.2.2. *The Bloch group $\mathcal{B}(\mathbb{C})$ is torsion-free.*

Relations between dilogarithm-values of elements in the Bloch group usually correspond to relations between these elements. For example, we have Borel's theorem [Bor77] that relates the vanishing of elements in the Bloch group to their values under the Bloch-Wigner dilogarithm. See [NY95, §2] for Borel's theorem stated in terms of the Bloch group.

Theorem 2.2.3 (Borel's theorem)**.** *Let $\xi \in \mathcal{B}(\mathbb{C})$ with algebraic coefficients. Then $\xi = 0$ if and only if $D(\sigma(\xi)) = 0$ for all complex embeddings $\sigma(\xi)$ of ξ. In this case, $L(\sigma(\xi)) \in 4\pi^2\mathbb{Q}$ for all real embeddings σ.*

It is expected that the converse of the last statement is also true. In other words, if $\xi \in \mathcal{B}(\mathbb{C})$ with $L(\xi) \in \pi^2\mathbb{Q}$, it is believed that ξ vanishes in $\mathcal{B}(\mathbb{C})$. Moreover, if $\xi \in \mathcal{B}(\mathbb{C})$ is totally real, then Borel's theorem implies $\xi = 0$.

Next, we will study explicit examples of elements in $\mathcal{B}(\mathbb{C})$.

2.2.1. Half-symplectic matrices

Examples of elements in the Bloch group come from solutions of certain equations associated to so-called half-symplectic matrices.

Definition 2.2.4. A matrix $(A|B) \in \mathbb{Z}^{r\times 2r}$ is *half symplectic* if it is the upper part of a symplectic matrix. In other words, if there exists $(C|D) \in \mathbb{Z}^{r\times 2r}$ such that $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbb{Z}^{2r\times 2r}$ is symplectic, i.e., if one of the following equivalent conditions is fulfilled

- the matrices A^TC and B^TD are symmetric and $A^TD - C^TB = 1$, or
- the matrices AB^T and CD^T are symmetric and $AD^T - BC^T = 1$.

The inverse of a symplectic matrix is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}. \tag{2.25}$$

We have the following equivalent characterisation of half-symplectic matrices that does not require the completion to a symplectic matrix. The equivalence follows immediately from the definition.

Proposition 2.2.5. *A matrix $(A|B) \in \mathbb{Z}^{r\times 2r}$ is half symplectic if and only if*

- *the matrix $A^T B$ is symmetric, and*
- *the $2r$ columns of $(A|B)$ span $\mathbb{Z}^r$ over $\mathbb{Z}$.*

From half-symplectic matrices we can construct elements in $\mathcal{B}(\mathbb{C})$.

Example 2.2.6. Let $(A|B) \in \mathbb{Z}^{r\times 2r}$ be a half-symplectic matrix. If $Q = (Q_1, \dots, Q_r) \in \mathbb{C}^r$ is a solution of the equations

$$\prod_{j=1}^{r}(1-Q_j)^{B_{ij}} \;=\; \prod_{j=1}^{r} Q_j^{A_{ij}}, \quad i = 1,\dots,r, \tag{2.26}$$

then $[Q] = \sum_{i=1}^{r}[Q_i]$ is an element in $\mathcal{B}(\mathbb{C})$. To see this, let $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathbb{Z}^{2r\times 2r}$ be the symplectic completion of $(A|B)$. We compute, using $A^T D - C^T B = I$,

$$\begin{aligned} d([Q]) &= \sum_{i=1}^{r} Q_i \wedge (1-Q_i) \\ &= \sum_{i,j=1}^{r} \left((A^T D)_{ij} - (C^T B)_{ij}\right)\, (Q_i \wedge (1-Q_j)) \\ &= \sum_{i,j=1}^{r} (A^T D)_{ij}\, (Q_i \wedge (1-Q_j)) - (C^T B)_{ij}\, (Q_i \wedge (1-Q_j)) \\ &= \sum_{k,j=1}^{r} D_{kj} \left(\prod_{i=1}^{r} Q_i^{A_{ki}} \wedge (1-Q_j)\right) - \sum_{k,i=1}^{r} C_{ki} \left(Q_i \wedge \prod_{j=1}^{r}(1-Q_j)^{B_{kj}}\right). \end{aligned} \tag{2.27}$$

We apply the relations in (2.26) to obtain

$$\begin{aligned} d([Q]) &= \sum_{k,j=1}^{r} D_{kj} \left(\left(\prod_{i=1}^{r}(1-Q_i)^{B_{ki}}\right) \wedge (1-Q_j)\right) - \sum_{k,i=1}^{r} C_{ki} \left(Q_i \wedge \left(\prod_{j=1}^{r} Q_j^{A_{kj}}\right)\right) \\ &= \sum_{i,k,j=1}^{r} D_{kj} B_{ki} ((1-Q_i)\wedge(1-Q_j)) - \sum_{j,k,i=1}^{r} C_{ki} A_{kj}\, (Q_i \wedge Q_j) \\ &= \sum_{i,j=1}^{r} (B^T D)_{ij} ((1-Q_i)\wedge(1-Q_j)) - \sum_{j,i=1}^{r} (A^T C)_{ji} (Q_i \wedge Q_j) \\ &= 0\,, \end{aligned} \tag{2.28}$$

because the matrices $B^T D$ and $A^T C$ are symmetric.

For a given half-symplectic matrix $(A|B)$, we will denote by $\mathcal{Q}_{(A|B)} \subset \mathbb{C}$ the set of solutions of (2.26).

The equations in (2.26) appear in the context of ideally triangulated hyperbolic 3–manifolds as gluing equations and are known as Neumann-Zagier equations [NZ85].

The most relevant case in this thesis will be the case where B is the identity matrix I_r. Then $(A|I_r)$ is half symplectic with symplectic completion $\left(\begin{smallmatrix} A & I_r \\ -I_r & 0\end{smallmatrix}\right)$ and the equations in (2.26) reduce to

$$1 - Q_i \;=\; \prod_{j=1}^{r} Q_j^{A_{ij}}, \qquad i = 1,\dots,r. \tag{2.29}$$

In this case, we write $\mathcal{Q}_A \subset \mathbb{C}^r$ for the set of solutions. These equation were discovered by Nahm in the context of Nahm sums, see below, and are thus called *Nahm equations*.

2.2.2. The extended Bloch group

Following [GZ07, GZ24, Neu04, Zic15], we introduce the extended Bloch group. Elements in the extended Bloch group will correspond to choices of $\log(z_j)$ and $\log(1-z_j)$ for $\sum_j [z_j] \in \mathcal{B}(\mathbb{C})$. For this, consider the abelian cover of $\mathbb{C}^\times \smallsetminus \{0,1\}$

$$\widehat{\mathbb{C}} \;=\; \{(u,v) \in \mathbb{C}^2 \mid e^u + e^v = 1\} \tag{2.30}$$

via $z = e^u = 1 - e^v$. As in (2.17), we define the map $\widehat{d} : \mathbb{Z}[\widehat{\mathbb{C}}] \to \wedge^2(\mathbb{C})$ and define the *extended Bloch group* as the quotient

$$\widehat{\mathcal{B}}(\mathbb{C}) \;=\; \ker \widehat{d}/\widehat{\mathcal{S}}, \tag{2.31}$$

where $\widehat{\mathcal{S}}$ is the subgroup generated by elements of the form $\sum_{j=1}^5 (-1)^j [u_j, v_j]$ for $(u,v) \in \widehat{\mathbb{C}}^5$ satisfying the lifted 5-term relation

$$(u_2,u_4) \;=\; (u_1+u_3, u_3+u_5), \qquad (v_1,v_3,v_5) \;=\; (u_5+v_2, v_2+v_4, u_1+v_4). \tag{2.32}$$

Elements $\widehat{\xi} = \sum_j [u_j, v_j] \in \widehat{\mathcal{B}}(\mathbb{C})$ in the extended Bloch group can be projected to elements $\xi = \sum_j [e^{u_j}] \in \mathcal{B}(\mathbb{C})$ in the Bloch group. Moreover, the *regulator map* is defined by

$$\sum_j [u_j, v_j] \mapsto \sum_j \mathcal{L}([u_j, v_j]) \in \mathbb{C}/4\pi^2\mathbb{Z}, \tag{2.33}$$

for $\sum_j [u_j, v_j] \in \widehat{\mathcal{B}}(\mathbb{C})$, where, with F defined in (2.7),

$$\mathcal{L}(u,v) \;=\; F(v) + \frac{1}{2}uv - \frac{\pi^2}{6}. \tag{2.34}$$

2.2.3. The volume

Let $(A|B)$ be a half-symplectic matrix as in Definition 2.2.4. Choose $\mu \in \frac{1}{2}\operatorname{diag}(B^TA) + \mathbb{Z}^r$ and let $Q \in \mathcal{Q}_{(A|B)} \subset \mathbb{C}^r$ be a solution of the equations (2.26) as in Example 2.2.6.

Proposition 2.2.7. *For Q, μ as above, let $u, v \in \mathbb{C}^r$ be choices of $(\log(Q_j))_j$, resp. $(\log(1-Q_j))_j$, such that $Au - Bv = 2\pi i\mu$. The* volume of Q *(with respect to μ) is defined by*

$$V(u,v) \;=\; \frac{1}{2}u^Tv + \frac{1}{2}(Au-Bv)^T(Cu-Dv) - r\frac{\pi^2}{6} + \sum_{j=1}^r F(v_j), \tag{2.35}$$

with F as in (2.7). Then $V(u,v)$ is a well-defined complex number modulo $4\pi^2\mathbb{Z}$. Moreover, $\operatorname{Im}(V(u,v))$ is a well-defined complex number and equals the Bloch-Wigner dilogarithm $\sum_j D(Q_j)$, where $Q_j = e^{u_j} = 1 - e^{v_j}$.

Proof. We prove that $V(u,v)$ does not change modulo $4\pi^2\mathbb{Z}$ if u, resp. v, changes by $2\pi i k$, resp. $2\pi i l$, for some $k, l \in \mathbb{Z}^r$. The condition $Au - Bv = 2\pi i\mu$ implies that $Ak = Bl$. We complete $(A|B)$ to a symplectic matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbb{Z}^{2r\times 2r}$ such that $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\left(\begin{smallmatrix} -k \\ l \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ n \end{smallmatrix}\right)$ for some $n \in \mathbb{Z}^r$. Using (2.25), we see that

$$\begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}\begin{pmatrix} 0 \\ n \end{pmatrix} \;=\; \begin{pmatrix} -k \\ l \end{pmatrix} \tag{2.36}$$

and thus $k = B^T n$ and $l = A^T n$. We compute

$$\begin{aligned}
&V(u + 2\pi i k, v + 2\pi i l) \\
&= V(u + 2\pi i B^T n, v + 2\pi i A^T n) \\
&= \frac{1}{2}u^T v + \frac{1}{2}2\pi i(n^T Bv + n^T Au) - 4\pi^2 n^T BA^T n \\
&\quad + \frac{1}{2}(Au - Bv + 2\pi i(AB^T - BA^T)n)^T(Cu - Dv + 2\pi i(CB^T - DA^T)n) \\
&\quad - r\frac{\pi^2}{6} + \sum_j F(v_j) - 2\pi i n^T A^T u \\
&= V(u, v) + \pi i n^T \left(Bv + Au - Au + Bv - 2Au + 2\pi i BA^T n\right) \\
&= V(u, v) - 4\pi^2 n^T \mu + 2\pi^2 n^T B^T An \\
&\in V(u, v) + 4\pi^2\mathbb{Z},
\end{aligned} \tag{2.37}$$

because $\mu \in \frac{1}{2}\operatorname{diag} AB^T + \mathbb{Z}^r$. This proves the claim. $\square$

The function V is called volume because it also appears as the hyperbolic volume of ideally triangulated hyperbolic 3-manifolds [NZ85]. The relation between the volume and the regulator map in equation (2.33) is given by

$$V(u, v) \;=\; \mathcal{L}([u, v]) + \frac{1}{2}(Au - v)^T(Cu - Bv). \tag{2.38}$$

If $B = I_r$ is the identity matrix, then $V(u, v)$ reduces to

$$V(u, v) \;=\; \frac{1}{2}u^T Au - r\frac{\pi^2}{6} + \sum_{j=1}^{r} F(v_j) \tag{2.39}$$

and is independent of the choice of v. In the context of the asymptotics of Nahm sums, we will encounter the principal branch of $V(u, v)$. Therefore, we write $V(Q) = V(u, v)$ where u, v are the principal branches of $\log(Q)$, resp. $\log(1 - Q)$. In particular, if $Q \in (0, 1)^r$, we have $V(Q) = \sum_j L(Q_j) - r\frac{\pi^2}{6}$, where L is the Rogers Dilogarithm, defined in (2.14).

2.3. The q-Pochhammer symbol and q-series identities

We define the *q-Pochhammer symbol*

$$(a; q)_n \;\coloneqq\; \prod_{i=0}^{n-1} 1 - aq^i \tag{2.40}$$

for $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. The identity

$$(a; q)_n \;=\; \frac{(a; q)_\infty}{(aq^n; q)_\infty} \tag{2.41}$$

suggests to define the q-Pochhammer symbol at $-n$ by setting

$$(a; q)_{-n} \;=\; \frac{(a; q)_\infty}{(aq^{-n}; q)_\infty} \;=\; \frac{1}{(aq^{-n}; q)_n}. \tag{2.42}$$

A simple computation shows that we have the relation

$$(a;q)_{-n} \;=\; q^{\frac{1}{2}(n(n-1))}\frac{(-q/a)^n}{(q/a;q)_n}. \tag{2.43}$$

For later use, we also recall some basic q-series identities.

Proposition 2.3.1 ([Zag07, Prp. 2]). *For $x, q \in \mathbb{C}$ with $|x|, |q| < 1$ we have*

$$\begin{aligned} (x;q)_\infty^{-1} &= \sum_{n=0}^{\infty} \frac{x^n}{(q)_n}, \\ (x;q)_\infty &= \sum_{n=0}^{\infty} \frac{x^n q^{n(n-1)/2}}{(q)_n}. \end{aligned} \tag{2.44}$$

Proposition 2.3.2 (q-binomial theorem, [Zag07, (13)]). *For $|q| < 1$ and $m, n \in \mathbb{Z}_{\geq 0}$ we have*

$$\frac{1}{(q)_n(q)_m} \;=\; \sum_{r=0}^{\min(m,n)} \frac{q^{(m-r)(n-r)}}{(q)_r(q)_{m-r}(q)_{n-r}}. \tag{2.45}$$

2.4. Modular forms and functions

Modular forms are important objects in number theory and have many fascinating connections to other fields of mathematics, such as algebraic geometry, mathematical physics, knot theory and combinatorics. We will discuss some of these connections later in this thesis. The book [BVdGHZ08] gives a nice overview of the theory of modular forms.

In the following, we denote by $\mathbb{H} := \{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$ the upper half-plane. The group $\mathrm{SL}_2(\mathbb{Z}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2} : ad - bc = 1\}$ acts on $\mathbb{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau \;:=\; \frac{a\tau + b}{c\tau + d}. \tag{2.46}$$

The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the elements $S = \left(\begin{smallmatrix} 0 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Definition 2.4.1. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index and $\chi : \Gamma \to \mathbb{C}$ be a character. A function $f : \mathbb{H} \to \mathbb{C}$ is called *weakly modular* for Γ and χ of weight $k \in \mathbb{Z}$ if it fulfils the transformation

$$f\left(\frac{a\tau + b}{c\tau + b}\right) \;=\; \chi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)(c\tau + d)^k f(\tau) \tag{2.47}$$

for all $\tau \in \mathbb{H}$ and all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

Assume that f is holomorphic and weakly modular with subexponential growth at infinity, meaning that

$$\begin{aligned} f(x + iy) &= \mathrm{O}(e^{Cy}) \qquad \text{as } y \to \infty, \\ f(x + iy) &= \mathrm{O}(e^{C/y}) \qquad \text{as } y \to 0, \end{aligned} \tag{2.48}$$

for *all* $C > 0$. Then we say that f is a *modular form* of weight $k \in \mathbb{Z}$ for Γ and χ.

For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, it is enough to consider the transformation of f under the generators S and T of $\mathrm{SL}_2(\mathbb{Z})$.

For a given Γ, χ and k as above, the corresponding vector space of modular forms is finite dimensional ([Miy06, §2.5]). It follows from Liouville's theorem that there are no modular forms of weight 0.

Definition 2.4.2. A *modular function for* Γ *and* χ is a meromorphic, weakly modular function of weight $k = 0$ with exponential growth at infinity, meaning the asymptotics in (2.48) hold for *some* $C > 0$.

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index. This implies that $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$ for some $N \in \mathbb{Z}_{\geq 1}$. Therefore, if f is weakly modular for Γ, f is N-periodic , meaning $f(\tau + N) = f(\tau)$ for all $\tau \in \mathbb{H}$, and has a Fourier expansion

$$f(\tau) \;=\; \sum_{n \gg -\infty} a_n q^{n/N}, \qquad q = e^{2\pi i \tau} \tag{2.49}$$

with $a_n \in \mathbb{C}$. The Fourier coefficients a_n are usually of interest in other areas of mathematics.

Throughout, we say that a function f is *modular* whenever f is a modular function for some subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ of finite index and some character.

Via $\tau \mapsto q = e^{2\pi i \tau}$ for $\tau \in \mathbb{H}$, we can identify q-series that converge for $|q| < 1$ as functions on the upper half-plane and study their modular behaviour. For such a q-series $f(q)$, we use the convention that $\tilde{f}(\tau) = f(e^{2\pi i \tau})$ is the corresponding function on the upper half-plane.

2.4.1. Examples: theta functions and Dedekind eta-function

One of the most important examples of modular forms are theta functions. We will use the Jacobi theta function to obtain their modular behaviour. Here and throughout, we write $e(x)$ for $e^{2\pi i x}$ for $x \in \mathbb{C}$.

Proposition 2.4.3. *Define the* Jacobi theta function *for* $\tau \in \mathbb{H}$*,* $z \in \mathbb{C}$ *by*

$$\vartheta(\tau, z) \;\coloneqq\; \sum_{n \in \mathbb{Z}} (-x)^n q^{\frac{1}{2}n(n+1)}, \qquad q = e^{2\pi i \tau},\, x = e^{2\pi i z}. \tag{2.50}$$

Then $\vartheta(\tau, z)$ *satisfies the transformations*

$$\begin{aligned} \vartheta(\tau + 1, z) \;&=\; \vartheta(\tau, z), \\ \vartheta(\tau, z + 1) \;&=\; \vartheta(\tau, z), \\ \vartheta(\tau, z + \tau) \;&=\; -q^{-1} x^{-1} \vartheta(\tau, z), \\ e\left(\frac{-1}{8\tau} + \frac{z}{2\tau}\right) \vartheta\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) \;&=\; \tau^{\frac{1}{2}} e\left(\frac{z^2}{2\tau} - \frac{3}{8}\right) e\left(\frac{\tau}{8} + \frac{z}{2}\right) \vartheta(\tau, z). \end{aligned} \tag{2.51}$$

The transformations of the Jacobi theta function in (2.51) mean that $\theta(\tau, z)$ is essentially a *Jacobi form* ([EZ85]) of weight $\frac{1}{2}$ and index $\frac{1}{2}$. In combination with growth conditions, the transformations imply that $\vartheta(\tau, \alpha + \beta\tau)$ is a modular form for all $\alpha, \beta \in \mathbb{Q}$. We can relate the Jacobi theta function to the q-Pochhammer symbol defined in (2.40).

Theorem 2.4.4 (Jacobi triple product identity)**.** *For* $\tau \in \mathbb{H}$*,* $z \in \mathbb{C}$ *we have*

$$\vartheta(\tau, z) \;=\; (q; q)_\infty \, (\tfrac{1}{x}; q)_\infty \, (xq; q)_\infty, \qquad q = e^{2\pi i \tau},\; x = e^{2\pi i z}. \tag{2.52}$$

An elementary proof of the Jacobi triple product identity is given by Andrews [And65]. Replacing (x, q) by (q^{-1}, q^3) in the Jacobi triple product identity implies that the *Dedekind eta-function*, defined by

$$\eta(\tau) \;\coloneqq\; q^{1/24} (q; q)_\infty \;=\; \sum_{n \in \mathbb{Z}} (-1)^n q^{(6n+1)^2/24}, \quad q = e^{2\pi i \tau}, \tag{2.53}$$

is a modular form of weight $\frac{1}{2}$ for $\mathrm{SL}_2(\mathbb{Z})$ in the sense that

$$\eta(\tau+1) \;=\; e\left(\frac{-1}{24}\right)\eta(\tau), \qquad \eta\left(\frac{-1}{\tau}\right) \;=\; \sqrt{-i\tau}\,\eta(\tau). \tag{2.54}$$

Here, the branch of the square root is chosen such that $|\arg\sqrt{-i\tau}| < \frac{\pi}{2}$. More generally, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\eta\left(\frac{a\tau+b}{b\tau+c}\right) \;=\; \epsilon(a,b,c,d)(c\tau+d)^{\frac{1}{2}}\eta(\tau), \tag{2.55}$$

where

$$\epsilon(a,b,c,d) \;=\; \begin{cases} e\left(\frac{b}{24}\right) & \text{if } c=0, d=1, \\ e\left(\frac{a+d}{24c} - \frac{1}{2}s(d,c) - \frac{1}{8}\right) & \text{if } c>0, \end{cases} \tag{2.56}$$

with the Dedekind sum

$$s(d,c) \;=\; \sum_{n=1}^{c-1} \frac{n}{c}\left(\frac{dn}{c} - \left\lfloor \frac{dn}{c} \right\rfloor - \frac{1}{2}\right). \tag{2.57}$$

2.4.2. The asymptotics of modular forms and functions

The asymptotics of a modular function or form as $\tau \to 0$, resp. $q = e^{2\pi i\tau} \to 1$ can be obtained from its modular transformation. The following result is a slight generalisation of [VZ11, Lemma 3.1] where we allow that τ goes to 0 on any ray in the upper half-plane. The proof stays the same.

Proposition 2.4.5 ([VZ11, Lemma 3.1])**.** *If $f : \mathbb{H} \to \mathbb{C}$ is a modular form or function of weight $k \in \mathbb{Z}$ then as $h \to 0$ on a ray in the right half-plane*

$$f(e^{-h}) \;=\; e^{\alpha\pi^2/h}\left(a_0 h^{-k} + \mathrm{O}\left(h^N\right)\right) \tag{2.58}$$

for all $N \geq 0$ and some $\alpha \in \mathbb{Q}$ and $a_0 \in \mathbb{C}$.

2.4.3. Vector-valued modular forms

For a modular function for some subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, it can be helpful to consider it as a component of a vector-valued modular function for $\mathrm{SL}_2(\mathbb{Z})$.

We say that a holomophic function $F : \mathbb{H} \to \mathbb{C}^n$ is a *vector-valued modular function* for $\mathrm{SL}_2(\mathbb{Z})$ and a representation ρ of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}^n$ if

$$F\Big(\frac{a\tau+b}{c\tau+d}\Big) \;=\; \rho(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))F(\tau) \tag{2.59}$$

for all $\tau \in \mathbb{H}$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and each component fulfils the same growth conditions as a modular function. As for scalar-valued modular function, we only need to give the transformation under $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The most important example for vector-valued modular functions in this thesis comes from a specialisation of the Jacobi theta function.

Example 2.4.6. If we substitute $\tau \mapsto 5\tau$ and $z \mapsto -2\tau$, resp. $z \mapsto -\tau$, in (2.50), we obtain the functions

$$\begin{aligned} \theta_{5,1}(\tau) &:= q^{1/40}\vartheta(5\tau,-2\tau) = \sum_{n\equiv 1 \bmod 10} (-1)^{[n/10]} q^{n^2/40}, \\ \theta_{5,2}(\tau) &:= q^{9/40}\vartheta(5\tau,-1\tau) = \sum_{n\equiv 3 \bmod 10} (-1)^{[n/10]} q^{n^2/40}. \end{aligned} \tag{2.60}$$

The transformation in (2.51) implies that the vector-valued function

$$\theta_5(\tau) := \begin{pmatrix} \theta_{5,1}(\tau) \\ \theta_{5,2}(\tau) \end{pmatrix} \qquad \text{for } \tau \in \mathbb{H} \tag{2.61}$$

fulfils the transformations

$$\theta_5(\tau+1) = \begin{pmatrix} \zeta_{40} & 0 \\ 0 & \zeta_{40}^9 \end{pmatrix} \begin{pmatrix} \theta_{5,1}(\tau) \\ \theta_{5,2}(\tau) \end{pmatrix}, \qquad \theta_5\Big(-\frac{1}{\tau}\Big) = \sqrt{\frac{\tau}{i}}\,\frac{2}{\sqrt{5}} \begin{pmatrix} \sin\frac{2\pi}{5} & \sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & -\sin\frac{2\pi}{5} \end{pmatrix} \theta_5(\tau), \tag{2.62}$$

where $\zeta_{40} = e^{\pi i/20}$. In combination with (2.54), this implies that the vector-valued function

$$\tilde{F}_2(\tau) := \frac{\theta_5(\tau)}{\eta(\tau)} \tag{2.63}$$

is a vector-valued modular function for $\mathrm{SL}_2(\mathbb{Z})$, satisfying

$$\tilde{F}_2(\tau+1) = \begin{pmatrix} \zeta_{60}^{-1} & 0 \\ 0 & \zeta_{60}^{11} \end{pmatrix} \tilde{F}_2(\tau), \qquad \tilde{F}_2\Big(-\frac{1}{\tau}\Big) = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\frac{2\pi}{5} & \sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & -\sin\frac{2\pi}{5} \end{pmatrix} \tilde{F}_2(\tau), \tag{2.64}$$

where $\zeta_{60} = e^{\pi i/30}$. The name of $\tilde{F}_2$ will become clear in the context of Nahm sums below.

3. Nahm sums and Nahm's conjecture

3.1. The Rogers-Ramanujan functions

Recall from equation (1.1) that the *Rogers-Ramanujan functions* are defined by

$$G(q) \;=\; \sum_{n\geq 0} \frac{q^{n^2}}{(q)_n}, \qquad H(q) \;=\; \sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n}. \tag{3.1}$$

They fulfil the Rogers-Ramanujan identities

$$G(q) \;=\; \prod_{n=\pm 1 \pmod 5} (1-q^n)^{-1}, \qquad H(q) \;=\; \prod_{n=\pm 2 \pmod 5} (1-q^n)^{-1} \tag{3.2}$$

that were proved by Rogers in 1894 [Rog94] and later rediscovered by Ramanujan. Ramanujan proved the identities by showing that both sides in both equations in (3.2), with an additional parameter, fulfil the same recurrence relations. See [Hir17, §15] for a historical discussion.

The Jacobi triple product (Proposition 2.52) implies that the functions $q^{-1/60}G(q)$ and $q^{11/60}H(q)$ are modular functions for $\Gamma_0(5) = \{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c = 0 \pmod 5\}$ because they can be written as a quotient of a theta function and the Dedekind eta-function

$$\begin{aligned} q^{-1/60}G(q) \;&=\; \frac{q^{-1/60}}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}(-1)^n q^{(5n^2+n)/2} \;=\; \frac{\theta_{5,1}(\tau)}{\eta(\tau)}, \\ q^{11/60}H(q) \;&=\; \frac{q^{11/60}}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}(-1)^n q^{(5n^2+3n)/2} \;=\; \frac{\theta_{5,2}(\tau)}{\eta(\tau)}, \end{aligned} \tag{3.3}$$

where $q = e^{2\pi i\tau}$ and $\theta_{5,j}(\tau)$ is defined in (2.60). From Example 2.4.6, we know that the vector-valued function

$$\tilde{F}_2(\tau) \;:=\; \begin{pmatrix} q^{-1/60}G(q) \\ q^{11/60}H(q) \end{pmatrix}, \qquad \text{where } q = e^{2\pi i\tau}, \tag{3.4}$$

is a vector-valued modular function satisfying the transformations

$$\tilde{F}_2(\tau+1) \;=\; \begin{pmatrix} \zeta_{60}^{-1} & 0 \\ 0 & \zeta_{60}^{11} \end{pmatrix}\tilde{F}_2(\tau), \qquad \tilde{F}_2\Big(-\frac{1}{\tau}\Big) \;=\; \frac{2}{\sqrt{5}}\begin{pmatrix} \sin(\frac{2\pi}{5}) & \sin(\frac{\pi}{5}) \\ \sin(\frac{\pi}{5}) & -\sin(\frac{2\pi}{5}) \end{pmatrix}\tilde{F}_2(\tau), \tag{3.5}$$

where $\zeta_{60} = e^{\pi i/30}$. As discussed in Chapter 1, the Rogers-Ramanujan functions appear in numerous areas of mathematics, such as partitions, VOAs, and quantum knot invariants. In 2019, Zagier gave a lecture at ICTP in Trieste [Zag19] on the Rogers-Ramanujan functions, where he lists even more connections to other fields of mathematics.

The fact that $q^{-1/60}G(q)$ and $q^{11/60}H(q)$ are modular functions often has a meaning coming from these applications. For example, as characters of the *rational* (2, 5)-minimal model VOA, they are modular functions by Zhu's modularity theorem ([Zhu96], see also [Gan06, §5.3.5]). In the context of the tail of the coloured Jones polynomial, the fact that the (2, 5)-torus knot is a *rational* knot ensures that $\Phi_K(q) = H(q)$ is modular up to a rational power of q (cf. Theorem 9.1.3).

3.2. Nahm sums

In the context of conformal field theory, Nahm [Nah07] noticed that several q-series appearing there have a similar form as the Roger-Ramanujan functions. For example, the functions

$$\sum_{n\geq 0} \frac{q^{\frac{1}{2}n^2-\frac{1}{48}}}{(q)_n} \;=\; \frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})\eta(2\tau)}, \qquad \sum_{n\geq 0} \frac{q^{\frac{1}{2}n(n+n)+\frac{1}{24}}}{(q)_n} \;=\; \frac{\eta(2\tau)}{\eta(\tau)}, \tag{3.6}$$

where $q = e^{2\pi i\tau}$, have a similar shape and are also modular functions. Functions of this form and generalisations thereof are sometimes called fermionic forms in physics or basic q-hypergeometric series. Motivated by these examples, Nahm asked the question, which generalisations of the Rogers-Ramanujan functions are modular functions or forms. We make the following general definition but we will consider different restrictions in different sections.

Definition 3.2.1. Let $A \in \mathbb{Q}^{r\times r}$ be a symmetric, positive definite matrix, $b \in \mathbb{Q}^r$ and $c \in \mathbb{Q}$. The *Nahm sum* for A, b, and c is defined by

$$f_{A,b,c}(q) \;:=\; \sum_{n\in\mathbb{Z}^r_{\geq 0}} \frac{q^{\frac{1}{2}n^T An+b^T n+c}}{(q)_{n_1}\cdots(q)_{n_r}} \;\in\; \mathbb{Z}((q^{\frac{1}{N}})), \tag{3.7}$$

where N is the denominator of the quadratic forn $\frac{1}{2}n^T An + b^t n + c$.

As for the Rogers-Ramanujan functions, Nahm sums and their generalisations occur in several fields of mathematics. For example, as characters of VOAs [CMP16, MP12, AvEH23], as quantum knot invariants [AD11, GZ21, GZ23, GL15, Whe23], in the context of cluster algebras [Miz21], and as generating functions of partitions [Mei54, BMRS23]. In Part II of this thesis, we will study the *tail of the coloured Jones polynomial* for alternating knots. This q-series knot invariant has an explicit representation as a generalised Nahm sum. In Part III, we will discuss a Nahm-type sum $v_1(q)$ from Ramanujan's "lost" notebook [And84, p.57, (1.10)] that counts a class of partitions. Andrews [And86] noticed that the coefficients of $v_1(q)$ have an *almost* regular sign pattern. In Part III, we explain his observations and prove some of his conjectures.

Nahm made the fascinating observation that the modularity of $f_{A,b,c}(q)$ in τ, where $q = e^{2\pi i\tau}$, for a given matrix A as above and some b, c is related to the vanishing of the images of the solutions of the *Nahm equation*

$$1 - Q_i \;=\; \prod_{j=1}^{r} Q_j^{A_{i,j}}, \qquad i = 1, \cdots, r \tag{3.8}$$

in the Bloch group $\mathcal{B}(\mathbb{C})$. Recall that we have already seen equation (3.8) in (2.29) as a specialisation of Example 2.2.6. In particular, the element $\sum_j [Q_j]$ is in the Bloch group $\mathcal{B}(\mathbb{C})$. The motivation behind this connection comes from the asymptotics of Nahm sums, in which solutions of the Nahm equation and their dilogarithm values appear. Comparing this to the asymptotics of general modular forms or functions suggests that the modularity of a given Nahm sum and the vanishing of $[Q] \in \mathcal{B}(\mathbb{C})$ are connected. We discuss this in detail in Section 3.3 below.

In the applications mentioned above, the corresponding Nahm equation (3.8) usually plays a role as well. In the context of knot theory, equation (3.8) and its generalisations are the glueing equations for the knot complement of a hyperbolic knot. In the context of VOAs, equation (3.8) is related to the TBA equation and the fusion rules [NRT93, Ter94].

We remark that equation (3.8) can explicitly be solved in terms of hypergeometric functions, see [Vil11].

3.2.1. A functional equation for Nahm sums

Nahm sums fulfil the following functional equation. We often ommit the c in the index of a Nahm sum $f_{A,b,c}(q)$, because $f_{A,b,c}(q) = q^c f_{A,b}(q)$.

Proposition 3.2.2. *Let $A \in \mathbb{Q}^{r\times r}$ be as above. For $j = 1, \dots, r$, the Nahm sums for A fulfil the recursion*

$$f_{A,b}(q) - f_{A,b+e_j}(q) \;=\; q^{\frac{1}{2}A_{jj}+b_j} f_{A,b+Ae_j}(q) \tag{3.9}$$

where $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ denotes the vector with 0 on all but the j-th entry.

Proof. We denote the summands in (3.7) by

$$b_n(q) \;=\; \frac{q^{\frac{1}{2}n^TAn+b^Tn}}{(q)_{n_1}\cdots(q)_{n_r}}, \qquad n = (n_1, \cdots, n_r) \in \mathbb{Z}^r_{\geq 0}. \tag{3.10}$$

Then for $j = 1, \dots, r$, the quotient of two consecutive summands is given by

$$\frac{b_{n+e_j}(q)}{b_n(q)} \;=\; q^{\frac{1}{2}A_{jj}+b_j}\frac{q^{e_j^TA_jn}}{1-q^{n_j+1}}. \tag{3.11}$$

In other words, we have

$$b_{n+e_j}(q)(1-q^{n_j+1}) \;=\; q^{\frac{1}{2}A_{jj}+b_j}\, b_n(q)q^{e_j^TA_jn}. \tag{3.12}$$

Summing both sides over $n \geq 0$ gives

$$f_{A,b}(q) - f_{A,b+e_j}(q) \;=\; q^{\frac{1}{2}A_{jj}+b_j} f_{A,b+Ae_j}(q) \tag{3.13}$$

as claimed. □

The previous proposition implies that for integral matrices A, the $\mathbb{Z}[q^{\pm 1}]$-module

$$\mathcal{N}_{A,b_0+\mathbb{Z}^r} \;:=\; \langle f_{A,b}(q) : b \equiv b_0 \bmod \mathbb{Z}^r\rangle \tag{3.14}$$

is finitely generated.

3.2.2. Modular Nahm sums

Using the asymptotics of Nahm sums, Zagier [Zag07] classified all modular one-dimensional Nahm sums.

Theorem 3.2.3 ([Zag07])**.** *For $r = 1$, the Nahm sum $f_{A,b,c}(q)$ is only modular in the following 7 cases.*

A	b	c	$f_{A,b,c}(q)$
2	0	$-1/60$	$\theta_{5,1}(\tau)/\eta(\tau)$
	1	$11/60$	$\theta_{5,2}(\tau)/\eta(\tau)$
1	0	$-1/48$	$\eta(\tau)^2/\eta(\tau/2)\eta(2\tau)$
	$1/2$	$1/24$	$\eta(2\tau)/\eta(\tau)$
	$-1/2$	$1/24$	$\eta(2\tau)/\eta(\tau)$
$1/2$	0	$-1/40$	$\theta_{5,1}(\tau/4)\eta(2\tau)/\eta(\tau)\eta(4\tau)$
	$1/2$	$1/40$	$\theta_{5,2}(\tau/4)\eta(2\tau)/\eta(\tau)\eta(4\tau)$

Based on numerical experiments, Zagier [Zag07, Table 2 and 3] also gives several two- and three-dimensional matrices for which some corresponding Nahm sums seem modular. For two-dimensional Nahm sums, all examples from Zagier's table are proven to be modular functions [CRW23, VZ11, Wan22]. The modularity for several three-dimensional Nahm sums listed by Zagier are proven [MW24, Wan22]. For higher-dimensional sums, apart from infinite families, only a few modular Nahm sums are known.

3.2.3. Asymptotics of Nahm sums

We will use the asymptotics to compare Nahm sums with modular forms and functions. For this, we need the following statement, see for example [VZ11, Lemma 2.1].

Proposition 3.2.4. *Let $A \in \mathbb{Q}^{r\times r}$ be a positive definite, symmetric matrix. Then the Nahm equations* (3.8) *have a unique solution $Q^{(1)} \in \mathcal{Q}_A$ in $(0,1)^r$.*

Different versions of the following result about the asymptotics of Nahm sums can be found in [GZ21, Mei54, VZ11, Zag07]. We will prove a slight generalisation of the result in Theorem 4.3.1 below.

Theorem 3.2.5. *Let A, b, c be as above. Then as $h \searrow 0$,*

$$f_{A,b,c}(e^{-h}) \;\sim\; e^{V(Q^{(1)})/h}\; \Phi_{A,b,c}^{(Q^{(1)})}(h) \tag{3.15}$$

where

1. *$Q^{(1)} \in \mathcal{Q}_A$ is the unique solution of* (3.8) *in* $(0,1)^r$,
2. *$\Phi_{A,b,c}^{(Q^{(1)})}(h) \in \overline{\mathbb{Q}}[[h]]$ is an explicit power series with algebraic coefficients, and*
3. *$V(Q^{(1)})$ denotes the volume of $Q^{(1)}$, defined after Proposition 2.2.7.*

We sketch the proof because it explains how the solutions of the Nahm equation (3.8) appear.

Idea of the proof. We expand the summands

$$b_n(q) \;=\; \frac{q^{\frac{1}{2}n^TAn+b^Tn}}{(q)_{n_1}\cdots(q)_{n_r}}, \qquad n=(n_1,\cdots,n_r)\in\mathbb{Z}^r_{\geq 0} \tag{3.16}$$

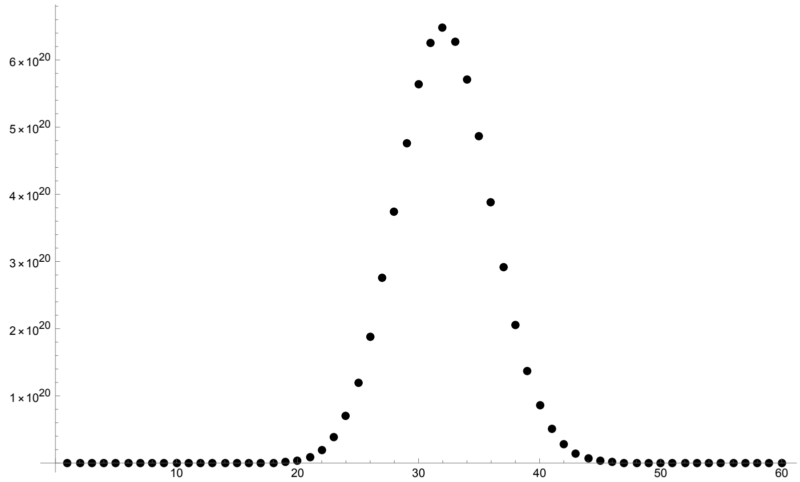


Figure 3.1.: $b_n(q)$ from (3.16) for $A = 4, b = 0$, and $h = \frac{1}{100}$.

around their maximum. Heuristically, the summands $b_n(q)$ are maximised when

$$1 \;\approx\; \frac{b_{n+e_j}(q)}{b_n(q)} \;=\; q^{\frac{1}{2}A_{jj}+b_j}\frac{q^{e_j^T A_j n}}{1-q^{n+1}}, \qquad \text{for } j = 1, \ldots, r. \tag{3.17}$$

With $Q = (q^{n_1}, \ldots, q^{n_r})$ and $q \to 1$ the previous equation can be written as

$$1 - Q_j \;\approx\; \prod_{k=1}^{r} Q_k^{A_{kj}}, \qquad \text{for } j = 1, \ldots, r. \tag{3.18}$$

Since $q = e^{-h} \in \mathbb{R}$, the summands $b_n(q)$ are maximised for $q^{n_j} = Q_j$, $j = 1, \ldots, r$, where Q is the unique solution of (3.8) in $(0,1)^r$. In other words, we have $n = -\frac{\log(Q)}{h}$.

The values $b_n(q)$ for $A = 4$, $b = 0$, $h = \frac{1}{100}$, and $n \leq 60$ are plotted in Figure 4.1. There, the summands $b_n(q)$ are maximised for $n = -\frac{\log(Q)}{h} = 32.22846\cdots$, where $Q = 0.7244920\cdots$ with $1 - Q = Q^4$.

□

Remark 3.2.6. *The quotient taken in* (3.17) *is the same as the one in* (3.11)*, which leads to the functional equation in Proposition 3.2.2. Therefore, we can guess the equation for maxima of the summands, and thus the corresponding Nahm equation, from a given functional equation: For each summand, the shift in b_j by* 1 *corresponds to a term Q_j.*

3.3. Nahm's conjecture

We will use the asymptotics from Theorem 3.2.5 to motivate Nahm's conjecture. Recall from Proposition 2.4.5 that if a function $f(q)$ is modular in τ of weight $k \in \mathbb{Z}$, where $q = e^{2\pi i \tau}$, then the asymptotics as $\tau = \frac{ih}{2\pi} \to 0$ has the special form

$$f(e^{-h}) \sim a_0 h^{-k} e^{\alpha/h}, \tag{3.19}$$

for some $a_0 \in \mathbb{C}$ and $\alpha \in 4\pi^2\mathbb{Q}$. Hence, if $f = f_{A,b,c}$ is a Nahm sum, we conclude

1. $k = 0$, i.e., f is a modular function,

2. the power series $\Phi_{A,b,c}^{(Q^{(1)})}(h)$ is constant, and

3. $V(Q^{(1)}) \in 4\pi^2\mathbb{Q}$.

From Section 2.2 we know that the last condition suggests the vanishing of $[Q^{(1)}]$ in $\mathcal{B}(\mathbb{C})$. This provides evidence that the modularity of $f_{A,b,c}$ is connected to the vanishing of $[Q^{(1)}]$ in $\mathcal{B}(\mathbb{C})$. Indeed, the following result, which was part of Nahm's original conjecture, was recently proven by Calegari-Garoufalidis-Zagier.

Theorem 3.3.1 ([CGZ23])**.** *If $f_{A,b,c}$ is modular, then $[Q^{(1)}]$ vanishes in $\mathcal{B}(\mathbb{C})$.*

In the proof, Calegari-Garoufalidis-Zagier use the asymptotics of Nahm sums $f_{A,b}(q)$ as q approaches a root of unity as well as results from algebraic K-theory.

On the other hand, the vanishing of $[Q^{(1)}]$ is not sufficient for the modularity of $f_{A,b,c}(q)$, as the following examples show.

Example 3.3.2. 1. Let $A = \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$. Then, the solutions of the corresponding Nahm equation are of the form $1 - Q_1 = Q_1^4$, $(1 - Q_2)^4 = Q_2$. In other words, $Q_1, 1 - Q_2 \in \mathcal{Q}_4$. In particular, if $Q_1 = 1 - Q_2 = 0.7244920\cdots$ is the unique solution in $(0, 1)$, then $[Q_1] + [1 - Q_1] = 0$, cf. Example 2.2.1. However, for other values of (Q_1, Q_2) with $Q_1 \neq 1 - Q_2$, we have $[Q_1] + [Q_2] \neq 0$ and the corresponding Nahm sum

$$f_{A,b,c}(q) = q^c f_{4,b_1}(q) f_{1/4,b_2}(q) \tag{3.20}$$

is not modular for any $b \in \mathbb{Q}^2, c \in \mathbb{Q}$. This can be proven following the ideas from subsection 4.3.2.

2. We also give an integral, non-diagonal example. For $A = \left(\begin{smallmatrix} 4 & 3 \\ 3 & 3 \end{smallmatrix}\right)$, the Nahm equations

$$1 - Q_1 = Q_1^4 Q_2^3, \qquad 1 - Q_2 = Q_1^3 Q_2^3, \tag{3.21}$$

can be reduced to

$$0 = (4Q_1^2 - 2Q_1 - 1)(2Q_1^2 - 2Q_1 + 1), \qquad Q_2 = 2 - Q_1^{-1}, \tag{3.22}$$

with explicit solutions

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \in \left\{ \begin{pmatrix} \frac{1+\sqrt{5}}{4} \\ 3 - \sqrt{5} \end{pmatrix}, \begin{pmatrix} \frac{1-\sqrt{5}}{4} \\ 3 + \sqrt{5} \end{pmatrix}, \begin{pmatrix} \frac{1-i}{2} \\ 1 - i \end{pmatrix}, \begin{pmatrix} \frac{1+i}{2} \\ 1 + i \end{pmatrix} \right\}. \tag{3.23}$$

The images of $\left(\frac{1\pm\sqrt{5}}{4}, 3 \mp \sqrt{5}\right)^T$ vanish in $\mathcal{B}(\mathbb{C})$ because they are totally real, while the images of $\left(\frac{1\pm i}{2}, 1 \pm i\right)^T$ do not because their images under the Bloch-Wigner dilogarithm do not vanish (Theorem 2.2.3). Moreover, there is no $b \in \mathbb{Q}^2, c \in \mathbb{Q}$ such that $f_{A,b,c}(q)$ is modular. We will prove this statement in subsection 4.3.2 using the asymptotics of Nahm sums on rays in the upper half-plane.

More examples of matrices in $\mathbb{Z}^{2\times 2}$ such that $[Q^{(1)}]$ vanishes in $\mathcal{B}(\mathbb{C})$ but the images of other solutions do not are listed in [Zag07, p.46]. We will discuss some of these examples in Section 6.3 below.

These observations motivated Zagier to state the following conjecture.

Zagier's version of Nahm's conjecture 3.3.3. *For a given $A \in \mathbb{Q}^{r\times r}$ as above, the modularity of $f_{A,b,c}(q)$ for some $b \in \mathbb{Q}^r, c \in \mathbb{Q}$ is equivalent to the vanishing of $[Q]$ for all solutions $Q \in \mathcal{Q}_A$ of the Nahm equation* (3.8).

As one can show ([Zag07]), for $r = 1$, the images of the solutions of $1 - Q = Q^A$ vanish in $\mathcal{B}(\mathbb{C})$ if and only if $A \in \{\frac{1}{2}, 1, 2\}$. Hence, Zagier's classification in Theorem 3.2.3 proves this conjecture for $r = 1$.

This conjecture is sometimes referred to as "Nahm's conjecture". It should be noted, however, that Nahm never made this conjecture. He only conjectured that modularity of Nahm sums cannot occur unless the unique solution in $(0, 1)^r$ of the Nahm equation vanishes in $\mathcal{B}(\mathbb{C})$ (as was confirmed by Theorem 3.3.1 stated above). Prediction 3.3.3 was stated by Zagier in [Zag07] as a potential precise formulation of Nahm's insight.

In [VZ11], Vlasenko and Zwegers gave counterexamples to this conjecture: There exist matrices $A \in \mathbb{Q}^{r\times r}$ (and even $A \in \mathbb{Z}^{r\times r}$) such that $f_{A,b,c}(q)$ is modular for some $b \in \mathbb{Q}^r$, $c \in \mathbb{Q}$ even though $[Q] \neq 0$ for some $Q \in \mathcal{Q}_A$. In other words, only some solutions $Q \in \mathcal{Q}_A$ seem to be important for the modularity of $f_{A,b,c}$. We will discuss these examples in Section 6.2.

It is still believed that the converse is true: The vanishing of $[Q]$ in $\mathcal{B}(\mathbb{C})$ for all $Q \in \mathcal{Q}_A$ is sufficient for the modularity of $f_{A,b,c}(q)$ for some $b \in \mathbb{Q}^r, c \in \mathbb{Q}$.

Even though Theorem 3.2.3 seems to solve the problem for $r = 1$, it is not completely enlightening for which b's and c's the Nahm sum $f_{A,b,c}$ is modular. Furthermore, it only claims modularity for *some* congruence subgroup. It turns out to be helpful to consider $f_{A,b,c}$ as a component of a vector-valued modular function for $\mathrm{SL}_2(\mathbb{Z})$. In Chapter 5, we will study the vector-valued modular behaviour of modular Nahm sums. In particular, we will see that the solutions $\mathcal{Q}_A$ will not only determine whether $f_{A,b,c}$ is modular, but also for which b's and c's the Nahm sum is modular and how these functions transform as a vector-valued modular function under $\mathrm{SL}_2(\mathbb{Z})$.

3.4. Nahm sums for half-symplectic matrices

Nahm sums as defined in (3.7) can be generalised to Nahm sums for half-symplectic matrices, cf. Definitione 2.2.4. They are also of great interest and appear for example in the context of knot invariants [GZ24]. For a half-symplectic matrix $(A|B) \in \mathbb{Z}^{r\times 2r}$ with $\det B \neq 0$ and a vector $b \in \mathbb{Q}^r$, we define the generalised Nahm sum by

$$f_{(A|B),b}(q) \;\coloneqq\; \sum_{n\in\mathbb{Z}^r} \frac{q^{\frac{1}{2}n^T AB^T n + b^T Bn}}{(q)_{(B^T n)_1}\cdots(q)_{(B^T n)_r}}. \tag{3.24}$$

The corresponding Nahm equation is then given by

$$\prod_{j=1}^{r}(1 - Q_j)^{B_{i,j}} \;=\; \prod_{j=1}^{r} Q_j^{A_{i,j}}, \quad i = 1, \ldots, r \tag{3.25}$$

as in (2.26). The Nahm sums defined in (3.7) correspond to $B = I_r$. If $B \in \mathbb{Z}^{r\times r}$ is invertible over $\mathbb{Z}$, we obtain $f_{(A|B),b}(q) = f_{B^{-1}A,B^{-1}b}(q)$ by substituting $n \mapsto B^{-T}n$ in the summation.

3.4.1. Relations between Nahm sums

We present an identity between Nahm sums for half-symplectic matrices such that the solutions of the corresponding Nahm equations (3.25) have the same image in $\mathcal{B}(\mathbb{C})$. This provides further evidence that Nahm sums and the Bloch group are closely related. This result has been discovered for Nahm sums with $B = I_r$ independendly by Zwegers and Ontiveros in unpublished work. See also [Zag20, Lecture 7] as well as [DG13] for similar results.

Proposition 3.4.1. *For a half-symplectic matrix $(A|B) \in \mathbb{Z}^{r\times 2r}$ write $A = (A_1|A_2|A')$, where A_1, A_2 denote the first two columns of A and similarly $B = (B_1|B_2|B')$. We define the matrices*

$$\tilde{A} = \begin{pmatrix} 0 & -1 & -1 & 0 \\ A_1 + A_2 & A_1 + B_2 & A_2 + B_1 & A' \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & B_1 & B_2 & B' \end{pmatrix}. \tag{3.26}$$

Then $(\widetilde{A}|\widetilde{B}) \in \mathbb{Z}^{(r+1)\times(2r+2)}$ is also a half-symplectic matrix and

$$f_{(A|B),b}(q) = f_{(\widetilde{A}|\widetilde{B}),\tilde{b}}(q) \tag{3.27}$$

where $\tilde{b} = (0, b)^T$. Moreover, $z = (z_1, \ldots, z_r) \in \mathbb{C}^r$ satisfies the Nahm equation for $(A|B)$ if and only if

$$\tilde{z} = \left(z_1 + z_2 - z_1z_2,\ \frac{z_1}{z_1 + z_2 - z_1z_2},\ \frac{z_2}{z_1 + z_2 - z_1z_2},\ z_3,\ \ldots, z_r\right) \in \mathbb{C}^{r+2} \tag{3.28}$$

satisfies the Nahm equation for $(\widetilde{A}|\widetilde{B})$.

By relation (2.24) in the Bloch group, we have $[z] = [\tilde{z}] \in \mathcal{B}(\mathbb{C})$.

If we apply the previous proposition to Nahm sums with $B = I_r$, we obtain the following corollary for Nahm sums as defined in (3.7).

Corollary 3.4.2. *For $A \in \mathbb{Q}^{r\times r}$ symmetric and positive definite and $b \in \mathbb{Q}^r$, set*

$$\begin{aligned} \bar{A} &= \begin{pmatrix} A_{11} + 2A_{12} + A_{22} & (A_1 + A_2)^T \\ A_1 + A_2 & A + e_{1,2} + e_{2,1} \end{pmatrix}, \\ \bar{b} &= (b_1 + b_2, b_1, \ldots, b_n)^T. \end{aligned} \tag{3.29}$$

Then we have

$$f_{A,b}(q) = f_{\bar{A},\bar{b}}(q). \tag{3.30}$$

Moreover, if $z \in \mathbb{C}^r$ satisfies the Nahm equation for A, then $\tilde{z}$ as in (3.28) satisfies the Nahm equation for $\bar{A}$.

Proof of Proposition 3.4.1. It can easily be checked that $\tilde{A}\tilde{B}^T = \tilde{B}\tilde{A}^T$. Moreover, the columns of $(\tilde{A}|\tilde{B})$ span $\mathbb{Z}^{r+1}$ over $\mathbb{Z}$ if the columns of $(A|B)$ span $\mathbb{Z}^r$ over $\mathbb{Z}$. Hence, $(\tilde{A}|\tilde{B})$ is half symplectic.

We apply the q-binomial theorem (Proposition 2.3.2)

$$\frac{1}{(q)_{n_1}(q)_{n_2}} = \sum_{r+t=n_1,\, s=t=n_2} \frac{q^{rs}}{(q)_r(q)_s(q)_t} \qquad \text{for all } n_1, n_2 \in \mathbb{Z} \tag{3.31}$$

to to first two q-Pochhammer symbols to obtain

$$\begin{aligned} f_{(A|B),b}(q) &= \sum_{n\in\mathbb{Z}^r} \frac{q^{\frac{1}{2}n^TAB^Tn+b^Tn}}{(q)_{(B^Tn)_1}\cdots(q)_{(B^Tn)_r}} \\ &= \sum_{n\in\mathbb{Z}^r} \sum_{\substack{r=(B^Tn)_1-t,\\ s=(B^Tn)_2-t}} \frac{q^{\frac{1}{2}n^tAB^tn+b^tn+rs}}{(q)_r(q)_s(q)_t(q)_{(B^Tn)_3}\cdots(q)_{(B^Tn)_r}} \\ &= \sum_{n\in\mathbb{Z}^r,t\in\mathbb{Z}} \frac{q^{\frac{1}{2}n^TAB^Tn+b^Tn+((B^Tn)_1-t)((B^Tn)_2-t)}}{(q)_{(B^Tn)_1-t}(q)_{(B^Tn)_2-t}(q)_t(q)_{(B^Tn)_r}\cdots(q)_{(B^Tn)_r}}. \end{aligned} \tag{3.32}$$

If we write $m = \left(\begin{smallmatrix} t \\ n \end{smallmatrix}\right) \in \mathbb{Z}^{r+1}$, we have

$$\begin{aligned} &\frac{1}{2}n^TAB^Tn + b^Tn + ((B^Tn)_1 - t)((B^Tn)_2 - t) \\ &= \frac{1}{2}m^T\left(\begin{pmatrix} 0 & 0 \\ 0 & AB^T \end{pmatrix} + \begin{pmatrix} 2 & (-B_1-B_2)^T \\ -B_1-B_2 & B_1^TB_2 + B_2^TB_1 \end{pmatrix}\right)m \\ &= \frac{1}{2}m^T\left(\begin{pmatrix} 0 & 0 \\ A_1+A_2 & A \end{pmatrix} + \begin{pmatrix} 0 & -1 & -1 & 0 & \dots & 0 \\ 0 & B_2 & B_1 & 0 & \dots & 0 \end{pmatrix}\right)\begin{pmatrix} 1 & 0 \\ -1 & B_1^T \\ -1 & B_2^T \\ 0 & B'^T \end{pmatrix}m \\ &= \frac{1}{2}m^T\widetilde{A}\widetilde{B}^Tm. \end{aligned} \tag{3.33}$$

Hence, we have

$$f_{(A|B),b}(q) = \sum_m \frac{q^{m^T\widetilde{A}\widetilde{B}^Tm/2+bm}}{(q)_{(\widetilde{B}^Tm)_1}\cdots(q)_{(\widetilde{B}^Tm)_r}} = f_{(\widetilde{A}|\widetilde{B}),\widetilde{b}}(q) \tag{3.34}$$

as claimed.

If $\widetilde{z} = (\widetilde{z}_1,\dots,\widetilde{z}_r)$ is given as in (3.28), then

$$z = \Big(1 - \frac{1-\widetilde{z}_1}{\widetilde{z}_2},\ 1 - \frac{1-\widetilde{z}_2}{\widetilde{z}_1},\ \widetilde{z}_3,\dots,\widetilde{z}_r\Big) = \Big(\widetilde{z}_1\widetilde{z}_3,\ \widetilde{z}_1\widetilde{z}_3,\ \widetilde{z}_3,\dots,\widetilde{z}_r\Big). \tag{3.35}$$

In particular, z fulfils (3.25) for $(A|B)$ if

$$\Big(\frac{1-\widetilde{z}_1}{\widetilde{z}_2}\Big)^{B_{i1}}\Big(\frac{1-\widetilde{z}_2}{\widetilde{z}_1}\Big)^{B_{i2}}\prod_{j=3}^r(1-\widetilde{z}_i)^{B_{ij}} = \big(\widetilde{z}_1\widetilde{z}_3\big)^{A_{i1}}\big(\widetilde{z}_2\widetilde{z}_3\big)^{A_{i2}}\prod_{j=3}^r\widetilde{z}_i^{A_{ij}}, \qquad i = 1,\dots,r. \tag{3.36}$$

Multiplying both sides by $\widetilde{z}_1^{B_{i1}}\widetilde{z}_2^{B_{i2}}$ gives

$$(1-\widetilde{z}_1)^{B_{i2}}(1-\widetilde{z}_2)^{B_{i1}}\prod_{j=3}^r(1-\widetilde{z}_i)^{B_{ij}} = \widetilde{z}_1^{A_{i1}+B_{i1}}\ \widetilde{z}_2^{A_{i2}+B_{i2}}\ \widetilde{z}_3^{A_{i1}+A_{i1}}\prod_{j=3}^r\widetilde{z}_i^{A_{ij}} \tag{3.37}$$

for $i = 1,\dots,r$. Moreover, we have

$$\widetilde{z}_3 = 1 - (1-z_1)(1-z_2) \tag{3.38}$$

such that $\widetilde{z}$ satisfies the Nahm equation (3.25) for $(\widetilde{A}|\widetilde{B})$. Since the argument also works the other way around, this completes the proof. $\square$

3.5. Generalisations of Nahm sums

We briefly discuss other generalisations of Nahm sums.

3.5.1. Mock theta functions

A few months before his death, Ramanujan wrote his last letter to Hardy, where he described a new class of functions he had found and called them "mock theta function". Without giving a definition, Ramanujan lists 17 examples, for instance,

$$f_0(q) \;=\; \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n}, \qquad f_1(q) \;=\; \sum_{n\geq 0} \frac{q^{n^2+n}}{(-q;q)_n}. \tag{3.39}$$

Nearly a century after Ramanujan's death, Zwegers [Zwe02] discovered in his Phd thesis that Ramanujan's mock theta functions are the holomorphic parts of harmonic Maass forms. This means that their completion by a non-holomorphic function transforms like a modular form [BFOR17, Zag09, Zwe02].

All of Ramanujan's mock theta functions are q-hypergeometric series and can be seen as modified Nahm sums. For example, the functions $f_0(q)$ and $f_1(q)$ from (3.39), can be seen as modified versions of the Rogers-Ramanujan functions (3.1) with $(q;q)_n$ replaced by $(-q;q)_n$. While the summation in (3.1) could be over $n \in \mathbb{Z}$, because $1/(q;q)_n = 0$ for $n < 0$, it is important to restrict the summation in (3.39) to $n \geq 0$ since $1/(-q;q)_n \neq 0$ for $n < 0$. The sums over $n < 0$ in (3.39) are the mock theta functions

$$\begin{aligned} \sum_{n<0} \frac{q^{n^2}}{(-q;q)_n} &= 2\sum_{k\geq 0} q^{(k+1)(k+2)/2}(-q;q)_k \;=\; 2\psi_0(q), \\ \sum_{n<0} \frac{q^{n^2+n}}{(-q;q)_n} &= 2\sum_{k\geq 0} q^{n(n+1)/2}(-q;q)_k \;=\; 2\psi_1(q), \end{aligned} \tag{3.40}$$

also listed by Ramanujan.

Theorem 3 in [Whe23] provides a connection between the mock theta functions f_0 and f_1 and the modular functions G and H by proving that the vector-valued function

$$g(x,q) \;=\; \frac{(xq;q)_\infty}{(q;q)_\infty} \left(\sum_{n\in\mathbb{Z}} \frac{q^{n^2-\frac{1}{60}} x^{2n}}{(xq;q)_n}, \sum_{n\in\mathbb{Z}} \frac{q^{n^2+n+\frac{11}{60}} x^{2n+1}}{(qx;q)_n} \right)^T \tag{3.41}$$

is a Jacobi form. By restricting $g(x,q)$ to $x = 1$, we obtain the Rogers-Ramanujan functions $f_{2,0,-1/60}(q), f_{2,1,11/60}(q)$. The restriction $x = -1$ yields a modular combination of the mock theta functions

$$q^{-1/60}\frac{(-q;q)}{(q;q)}((f_0(q) + 2\phi_0(q)), \qquad q^{11/60}\frac{(-q;q)}{(q;q)}(f_1(q) + 2\phi_1(q))), \tag{3.42}$$

cf. [Wat37, p.279]. A similar construction seems to work for all mock theta functions listed by Ramanujan, provided that the sum over the non-positive integers converges. If the sum does not converge, [Whe23, Chapter 7] suggests a resummation that preserves the modularity.

The similarities between modular Nahm sums and Ramanujan's mock theta functions suggest that the modularity of mock theta functions is also related to the vanishing of elements in the Bloch group, and it would be desirable to study this connection further.

3.5.2. Nahm sums for symmetrisable matrices

In their PhD studies, Kanade and Russel numerically found several sum-product identities between q-series generalising the Rogers-Ramanujan identities (1.2). For example, the "modulo 9 conjecture" states that

$$\sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2}}{(q;q)_m(q^3;q^3)_n} \stackrel{?}{=} (q,q^3,q^6,q^8;q^9)_\infty^{-1}. \tag{3.43}$$

While most of their conjectures have been proven, the "modulo 9 conjecture" is still open. Motivated by these (conjectured) identities and examples coming from cluster algebras [Miz21], Mizuno [Miz23] studies *Nahm sums for symmetrisable matrices* of the form

$$\sum_{n\in\mathbb{Z}^r_{\geq 0}} \frac{q^{\frac{1}{2}n^T A\,\mathrm{diag}(d)n+b^Tn}}{(q^{d_1};q^{d_1})_{n_1}\cdots(q^{d_r};q^{d_r})_{n_r}} \tag{3.44}$$

where $A\in\mathbb{Z}^{r\times r}, d\in\mathbb{Z}^r, b\in\mathbb{Q}^r$ such that $A\,\mathrm{diag}(d)$ is symmetric and positive definite. As for ordinary Nahm sums, we can associate a generalised Nahm equation to symmetrisable matrices. Then the modularity of Nahm sums for symmetrisable matrices is conjectured to be related by the vanishing of the classes of these solutions in the Bloch group. For example, Mizuno [Miz23] proves a theorem similar to Theorem 3.3.1.

The following proposition shows that Nahm sums for symmetrisable matrices can be reduced to Nahm sums as defined in (3.7). The statement generalises Theorem 4.1 in [VZ11], and the proof uses the following identity from there: For all $n,m\in\mathbb{Z}_{\geq 0}$ we have

$$\frac{1}{(q;q)_n} = \sum_{\substack{k\in\mathbb{Z}^m_{\geq 0}\\ k_1+\cdots+k_m=n}} \frac{q^{\frac{m}{2}k^Tk+ml(m)^Tk}}{(q^m;q^m)_{k_1}\cdots(q^m;q^m)_{k_m}} \tag{3.45}$$

where $l(m)=(\frac{2i-m-1}{2m})_{i=1,\ldots,m}\in\mathbb{Q}^m$.

Proposition 3.5.1. *Let A,d,b be as above and $f\in\mathbb{Z}_{\geq 1}$ a common multiple of d. We set*

$$m_j = \frac{f}{d_j}\in\mathbb{Z}_{\geq 1}, \qquad r' = \sum_{i=1}^r m_i. \tag{3.46}$$

Moreover, define the matrix

$$A' = I_{r'} + \frac{1}{f}\mathcal{M}(A)\in\mathbb{Z}^{r'\times r'}, \tag{3.47}$$

where $\mathcal{M}(A)=(\mathcal{M}(A)_{i,j})_{i,j=1,\ldots,r}$ is a block matrix of the form

$$\mathcal{M}(A)_{i,j} = E_{m_i\times m_j}((A\,\mathrm{diag}(d))_{ij}-\delta_{i,j}d_i)\in\mathbb{Z}^{m_i\times m_j}. \tag{3.48}$$

Here, $E_{m_i\times m_j}$ denotes the matrix of size $m_i\times m_j$ with 1's in every entry. Moreover, set $b'=(b'_j)_{j=1,\cdots,r}$ with $b'_j\in\mathbb{Q}^{m_j}$ given by $b'_j = b_jE_{1\times m_j}+l(m_j)$. Then

$$\sum_{n\in\mathbb{Z}^r_{\geq 0}} \frac{q^{\frac{1}{2}n^T A\,\mathrm{diag}(d)n+b^Tn}}{(q^{d_1};q^{d_1})_{n_1}\cdots(q^{d_r};q^{d_r})_{n_r}} = f_{A',b'}(q^f). \tag{3.49}$$

Proof. We apply identity (3.45) to each Pochhammer symbol $(q^{d_j}; q^{d_j})_{n_j}$ for $j = 1, \dots, r$ in the left-hand side of (3.49) with $q \mapsto q^{d_j}$, $m \mapsto m_j$. Then we obtain that the sum is equal to

$$\sum_{n \in \mathbb{Z}_{\geq 0}^r} q^{\frac{1}{2} n^T A \,\mathrm{diag}(d) n + b^T n} \prod_{j=1}^{r} q^{-\frac{d_j}{2} n_j^2} \sum_{\substack{k^{(j)} \in \mathbb{Z}_{\geq 0}^{m_j} \\ \sum_l k_l^{(j)} = n_j}} \frac{q^{\frac{m_j}{2} d_j (k^{(j)})^T k^{(j)} + e_j l_{e_j}^T k^{(j)}}}{(q^f; q^f)_{k_1^{(j)}} \cdots (q^f; q^f)_{k_{m_j}^{(j)}}}. \tag{3.50}$$

We replace n_j by $\sum_l k_l^{(j)}$ in the first term and concatenate the vectors $k^{(j)}$ for $j = 1, \dots, r$ into a vector $k \in \mathbb{Z}^{r'}$. Then we obtain that the sum equals

$$\sum_{k \in \mathbb{Z}_{\geq 0}^{r'}} \frac{q^{\frac{1}{2} k^T \mathcal{M}(A) k + \frac{f}{2} k^T k + f b'^T k}}{(q^f; q^f)_{k_1} \cdots (q^f; q^f)_{k_{r'}}} \;=\; f_{A', b'}(q^f) \tag{3.51}$$

which completes the proof. □

Example 3.5.2. The sum in the "modulo 9 conjecture" in equation (3.43) can be written as a Nahm sum for symmetrisable matrices as in equation (3.44) with

$$A \;=\; \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \qquad d \;=\; (1,\, 3), \qquad b \;=\; (0,\, 0). \tag{3.52}$$

Applying Proposition 3.5.1 with $m = (3,\, 1)$, $f = 3$, we obtain with

$$\mathcal{M}(A)_{11} \;=\; \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathcal{M}(A)_{21} \;=\; \mathcal{M}(A)_{12}^T \;=\; \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}, \qquad \mathcal{M}(A)_{22} \;=\; (3), \tag{3.53}$$

that the sum-side in (3.43) is equal to $f_{A', b'}(q^3)$ where

$$A' \;=\; I_4 + \frac{1}{3} \begin{pmatrix} \mathcal{M}(A)_{11} & \mathcal{M}(A)_{12} \\ \mathcal{M}(A)_{21} & \mathcal{M}(A)_{22} \end{pmatrix} \;=\; \begin{pmatrix} \frac{4}{3} & \frac{1}{3} & \frac{1}{3} & 1 \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{4}{3} & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \qquad b' \;=\; \begin{pmatrix} -\frac{1}{3} \\ 0 \\ \frac{1}{3} \\ 0 \end{pmatrix}. \tag{3.54}$$

4. Asymptotics of Nahm sums

4.1. Introduction

We have seen in Section 3.3 that the asymptotics of Nahm sums motivate Nahm's observation that the modularity of Nahm sums is related to the vanishing of elements in the Bloch group. Theorem 3.2.5 gave the asymptotics of $f_{A,b}(e^{-h})$ as $h \searrow 0$. In this chapter, we will provide a slight generalisation of this result by considering the asymptotics of $f_{A,b}(e^{-h})$ for $h \to 0$ on any ray in the right half-plane. These asymptotics will be crucial for understanding "counterexamples" for Nahm's conjecture, as stated in [Zag07]. The proof of the generalised theorem will closely follow the proof of the known case. The idea of considering asymptotics on rays in the upper half-plane was motivated by similar asymptotic results by Garoufalidis-Zagier [GZ23].

4.2. Setup

We recall formal Gaussian integration and define power series that will appear in the asymptotics of Nahm sums.

4.2.1. Formal Gaussian integration

We define a formal version of Gaussian integration, see also [DG13, GSW23, GZ21]. If $\mathrm{Re}(a) > 0$, recall that the Gaussian integral evaluates to

$$\int_{\mathbb{R}} x^n e^{-\frac{1}{2}ax^2} dx \;=\; \begin{cases} a^{-\frac{n}{2}}(n-1)!! & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{4.1}$$

where $k!!$ for $k \in \mathbb{Z}_{\geq 0}$ denotes the *double factorial*, defined by

$$k!! \;:=\; \prod_{i=1}^{\lceil \frac{k}{2} \rceil - 1} (k-2i) \;=\; k(k-2)(k-4)\cdots(k-2\lceil \tfrac{k}{2} \rceil + 2). \tag{4.2}$$

If $\mathrm{Re}(a) \leq 0$, convergence of the integral is not guaranteed. However, (4.1) motivates to define the *formal Gaussian integral* for any $a \in \mathbb{C}$ and $n \in \mathbb{Z}$ by

$$\langle x^n \rangle_{x,a} \;:=\; \begin{cases} a^{-\frac{n}{2}}(n-1)!! & n \text{ even,} \\ 0 & n \text{ odd} \end{cases} \tag{4.3}$$

and $\langle f(x) \rangle_{x,a}$ via linearity for any polynomial $f(x) \in \mathbb{C}[x]$. Then $\langle x^n \rangle_{x,a} = \int_{\mathbb{R}} x^n e^{-\frac{1}{2}ax^2} dx$ if $\mathrm{Re}(a) > 0$.

We also define a higher-dimensional formal Gaussian integral for any polynomial in r variables $f(x) \in \mathbb{C}[x]$, $x = (x_1, \ldots, x_r)$ and any $A \in \mathbb{C}^{r \times r}$ with $\det A \neq 0$ via

$$\langle f(x) \rangle_{x,A} \;:=\; \sqrt{\frac{(2\pi)^n}{\det A}} \exp\left(\frac{1}{2} \sum_{i,j=1}^{r} (A^{-1})_{i,j} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right) f(x) \Bigg|_{x=0}. \tag{4.4}$$

Then

$$\langle f(x)\rangle_{x,A} \;=\; \int_{\mathbb{R}^r} f(x)e^{-\frac{1}{2}x^TAx}dx \tag{4.5}$$

if A is positive definite. The formal Gaussian integration fulfils the same basic properties as ordinary Gaussian integration, see e.g.,[GSW23].

Lemma 4.2.1. *1. For all invertible matrices $P\in\mathbb{C}^{r\times r}$, we have*

$$\langle f(Px)\rangle_{x,P^TAP} \;=\; \langle f(x)\rangle_{x,A}\,. \tag{4.6}$$

2. For $b\in\mathbb{C}$, we have

$$\langle \exp(-b^TAx)f(x+b)\rangle_{x,A} \;=\; \exp\left(\frac{1}{2}b^TAb\right)\langle f(x)\rangle_{x,A}. \tag{4.7}$$

4.2.2. Definition of $\Phi^{(Q)}_{A,b}(h)$

For a solution $Q\in\mathcal{Q}_A$ of the Nahm equation we define the integral

$$\widehat{\Phi}^{(Q)}_{A,b}(h) \;= (q;q)_\infty^{-r}\int \exp\Big(-\frac{h}{2}t^TAt - hb^Tt\Big)\prod_{j=1}^{r}(e^{-ht_j}q;q)_\infty dt \tag{4.8}$$

where the integral goes over a small neighbourhood of $t=-\log(Q)$. We remark that this resembles Nahm sums with the sum replaced by an integral. We will give an explicit description of $\Phi^{(Q)}_{A,b}(h)$ as a power series. Therefore, recall that the Bernoulli polynomials are defined for $p\in\mathbb{Z}_{\geq 0}$ by

$$B_p(x) \;=\; \sum_{k=0}^{p}\binom{p}{k}B_kx^{p-k}, \tag{4.9}$$

where $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6},\cdots$ are the Bernoulli numbers. We define for $t=(t_1,\ldots,t_r)$ the power series

$$\Psi^{(Q)}_{A,b}(t,h) \;=\; \sum_{i=1}^{r}\left(b_i+\frac{1}{2}\frac{Q_i}{1-Q_i}\right)t_ih^{1/2} - \sum_{p\geq 3}\frac{1}{p!}B_p\left(\frac{t_i}{h^{1/2}}\right)\mathrm{Li}_2(Q_i)h^{p-1}\in\mathbb{C}[[t,h^{1/2}]], \tag{4.10}$$

and a matrix

$$\widetilde{A}(Q) \;=\; A+\mathrm{diag}\Big(\frac{Q_j}{1-Q_j}\Big)_j\,, \tag{4.11}$$

then $\widehat{\Phi}^{(Q)}_{A,b}(h)$ has an explicit representation as power series.

Lemma 4.2.2. *With the power series*

$$\Phi^{(Q)}_{A,b}(h) \;=\; (2\pi)^{-r/2}\prod_{i=1}^{r}\frac{Q_i^{b_i}}{1-Q_i}\langle\exp(\Psi^{(Q)}_{A,b}(t,h))\rangle_{t,\tilde{A}(Q)} \;\in\; \mathbb{C}[[h]] \tag{4.12}$$

we have

$$\widehat{\Phi}^{(Q)}_{A,b}(h) \;=\; e^{V(Q)/h}\,\Phi^{(Q)}_{A,b}(h), \tag{4.13}$$

where $V(Q)$ denotes the principal branch of the volume defined after Proposition 2.2.7.

We will proof the Lemma alongside with Theorem 4.3.1 in subsection 4.4.3 by expanding the q-pochhamer symbol in (4.8).

For a fixed A, the power series $\Phi^{(Q)}_{A,b}(h)$ fulfil the same recursion as the corresponding Nahm sums in Proposition 3.2.2.

Proposition 4.2.3. *We have for $j = 1, \dots, r$*

$$\Phi^{(Q)}_{A,b}(h) - \Phi^{(Q)}_{A,b+e_j}(h) \;=\; e^{-(\frac{1}{2}A_{j,j}+b_j)h}\, \Phi^{(Q)}_{A,b+A_j}(h). \tag{4.14}$$

Proof. The proof can be seen as a continuous version of the proof of Proposition 3.2.2. In (4.8), we use the recursion

$$(e^{-ht_j}q;q)_\infty \;=\; (1-e^{-ht_j}q)(e^{-ht_j}q^2;q)_\infty \tag{4.15}$$

for the q-Pochhammer symbol. Then the substitution $t \mapsto t + e_j$ in the integral yields the claimed functional equation. ☐

One can also prove (4.14) using the representation in (4.10) and the symmetries of the Bernoulli polynomials.

The power series defined in (4.12) and generalisations thereof also appear in the context of topological invariants [DG13] and are believed to agree with the asymptotic expansion appearing in the Volume conjecture for the Kashaev invariant for knots, cf. Conjecture 7.2.8 in Part II of this thesis. In a paper with Garoufalidis and Wheeler [GSW23], we prove that these h-series fulfil relations that correspond to relations in the Bloch group for $Q \in \mathcal{Q}_A$. This implies that the series defined there are topological invariants, i.e., are independent of the chosen triangulation of a cusped hyperbolic 3-manifold. The identities proven there can be seen as an analogue for the h-series $\Phi^{(Q)}_{A,b}(h)$ of the identities in Proposition 3.4.1 for the q-series $f_{A,b}(q)$.

4.3. Asymptotics of Nahm sums on rays in the upper half-plane

The following is a generalisation of the asymptotics of Nahm sums in Theorem 3.2.5. It extends the asymptotics by allowing h to be on any ray in the right half-plane, and thus $\tau = \frac{ih}{2\pi}$ in the upper half-plane.

Theorem 4.3.1. *As $h \to 0$ on a ray in the right half-plane, we have*

$$f_{A,b}(e^{-h}) \;=\; \sum_{Q\in\mathcal{Q}_A} e^{V(Q)/h}\, \Phi^{(Q)}_{A,b}(h)\,(1+O(h^L)). \tag{4.16}$$

for all $L > 0$.

The asymptotics should be interpreted as a sum over several contributions where only the leading ones are relevant. In other words, only the contributions that maximise $|e^{V(Q)/h}|$ are leading should be taken into consideration. However, in order to emphasise that the subleading contributions do exist, but are just hidden, we include all solutions in the sum. In Section 5.2, we will see examples where the non-leading contributions become numerically visible.

For $\arg h = 0$, we obtain the asymptotics given in Theorem 3.2.5, corresponding to the unique solution $Q = Q^{(1)} \in \mathcal{Q}_A$ of (3.8) in $(0,1)^r$.

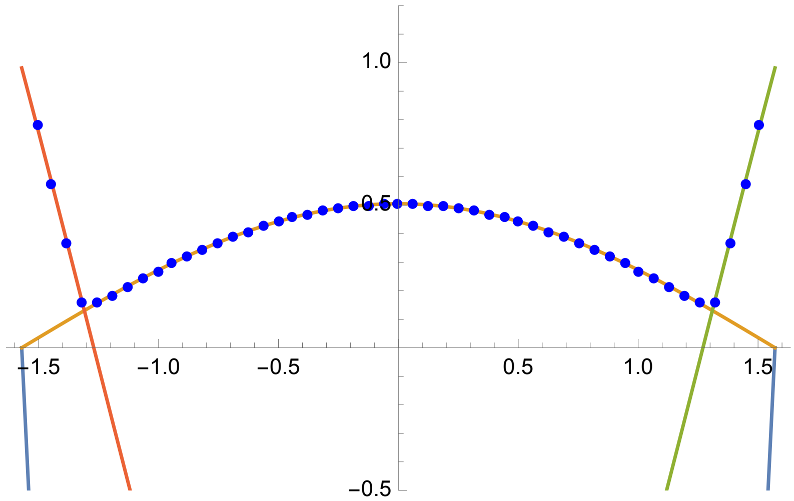


Figure 4.1.: equations (4.20) and (4.21) for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

4.3.1. Examples

Numeric computations for the asymptotics in Theorem 4.3.1 can be done with Zagier's asymptotic method [GMZ08, Zag22]. We illustrate the asymptotics in Theorem 4.3.1 with $A = 4$.

Example 4.3.2. Denote the solutions of (3.8) for $A = 4$ by

$$\mathcal{Q}_4 \;=\; \{Q \in \mathbb{C} : 1 - Q = Q^4\}. \tag{4.17}$$

We enumerate the solutions $\mathcal{Q}_4$ by

$$\begin{aligned} Q^{(1)} &= 0.7244920\cdots, \\ Q^{(2)} &= -1.220744\cdots, \\ Q^{(3)} &= 0.2481261\cdots - 1.033982\cdots i, \\ Q^{(4)} &= 0.2481261\cdots + 1.033982\cdots i \;=\; \overline{Q^{(3)}} \end{aligned} \tag{4.18}$$

with volumes

$$\begin{aligned} V_1 &= V(Q^{(1)}) = 0.5049814\cdots, \\ V_2 &= V(Q^{(2)}) = -17.20284\cdots, \\ V_3 &= V(Q^{(3)}) = -3.165607\cdots + 0.9813688\cdots i, \\ V_4 &= V(Q^{(4)}) = -3.165607\cdots - 0.9813688\cdots i. \end{aligned} \tag{4.19}$$

The values of

$$\varepsilon \log\left|f_{4,0}\left(e^{-he^{i\vartheta}}\right)\right| \tag{4.20}$$

for $\varepsilon = .001$ and several $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, as well as their asymptotic approximations

$$\mathrm{Re}\left(V_k e^{i\theta}\right), \quad k = 1, 2, 3, 4, \tag{4.21}$$

are plotted in Figure 4.1. The figure shows that for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ the expression (4.20) is asymptotically equal to the maximum of the 4 values in (4.21). The asymptotics depends on $\theta = \arg(h)$ and we distinguish three different cases:

1. If $|\theta| < \operatorname{atan}\frac{\operatorname{Re}(V_1-V_3)}{\operatorname{Im}(V_3)} = 1.309547\cdots$, the contribution for the unique solution $Q^{(1)} \in (0,1)$ is the leading one. With $\delta_4(Q^{(1)}) = 4 - 3Q^{(1)} = 1.826524\cdots$ the asymptotics is as in Theorem 3.2.5, namely
$$\begin{aligned} f_{4,0}(e^{-h}) &= e^{V_1/h}\frac{1}{\sqrt{\delta_4(Q^{(1)})}}\Bigg(1+\frac{-10304-31384Q^{(1)}+10422{Q^{(1)}}^2+26919{Q^{(1)}}^3}{961068}h \\ &\quad + \frac{-5211524+36879952Q^{(1)}-41646809{Q^{(1)}}^2-58173507{Q^{(1)}}^3}{6527573856}h^2+O(h^3)\Bigg) \\ &= e^{V_1/h}\left(0.7399243\cdots - 0.01334570\cdots h - 0.002547545\cdots h^2 + O(h^3)\right). \end{aligned} \tag{4.22}$$

2. If $\theta > 1.309547\cdots$, the contribution coming from $Q^{(3)}$ is the leading one and we have
$$\begin{aligned} f_{4,0}(e^{-h}) \;=\; e^{V_3/h}(&(0.4378251\cdots - 0.1751854\cdots i) \\ &+ (-0.01316992\cdots + 0.03260324\cdots i)h \\ &+ (0.004246714\cdots - 0.007143060\cdots i)h^2 + O(h^3)) \end{aligned} \tag{4.23}$$
and an algebraic formula as in equation (4.22) with $Q^{(1)}$ replaced by $Q^{(3)}$.

3. If $\theta < -1.309547\cdots$, the contribution corresponding to Q_4 is the leading one and the asymptotic expansion is the complex conjugate of the expansion in (4.23) or (4.22) with $Q^{(1)}$ replaced by $Q^{(4)}$.

The contribution for the negative solution $Q^{(2)}$ is never the leading one. However, one can imagine that the non-leading contributions still exist but are hidden behind the leading one. In Section 5.2, we will make them numerically visible.

4.3.2. Consequences

The refined asymptotics provide a new tool of proving the non-modularity of Nahm sums. Assume that f is a modular function. By Proposition 2.4.5, as $h \to 0$ along any ray in the right half-plane we have
$$f(e^{-h}) \;=\; e^{\alpha/h}(a + O(h^L)) \tag{4.24}$$
for all $L > 0$, some $a \in \mathbb{C}$, and some $\alpha \in \pi^2\mathbb{Q} \subset \mathbb{R}$. Let $g(q)$ be a function on the unit disc. Equation (4.24) implies that if $g(e^{-h}) = e^{V/h}(a + O(h))$ for some $a \in \mathbb{C}$ with $V \notin \mathbb{R}$ as $h \to 0$ on a ray in the right half-plane, then g cannot be modular. We will use this idea to prove non-modularity of some Nahm sums.

Example 4.3.3. In Example 4.3.2, we saw that for h on a ray in the right half-plane with $\arg h = .12\pi = 1.507964\cdots$ we have
$$f_{4,b}(e^{-h}) = e^{V_3/h}\frac{1}{\sqrt{4-3Q^{(3)}}}(1+\mathrm{O}(h)) \tag{4.25}$$
with $V_3 = -3.165607\cdots + 0.9813688\cdots i \notin \mathbb{R}$. Hence, $f_{4,b}(q)$ is not modular for any $b, c \in \mathbb{Q}$.

Recall from Proposition 2.2.7 that the imaginary part of $V(Q)$ equals the Bloch-Wigner dilogarithm of Q, which is related to the vanishing of $[Q] \in \mathcal{B}(\mathbb{C})$ by Theorem 2.2.3. This idea can be used to prove the following result.

Theorem 4.3.4. *Let $A \in \mathbb{Z}^{r\times r}$ be as above and assume that $V(Q) \neq V(Q')$ for all $Q, Q' \in \mathcal{Q}_A$. If $f_{A,b,c}(q)$ is modular for some $b \in \mathbb{Q}^r, c \in \mathbb{Q}$, then $[Q] = 0$ for all $Q \in \mathcal{Q}_A$.*

Proof. If $f_{A,b,c}(q)$ is modular, the asymptotic of $f_{A,b,c}(e^{-h})$ as $h \to 0$ on any ray in the right half-plane is given as in equation (4.24) for some $\alpha \in \pi^2\mathbb{Q} \subset \mathbb{R}$. Comparing this with the asymptotic in Theorem 4.3.1 implies that $\mathrm{Im}(V(Q)) = D(Q) = 0$ for all $Q \in \mathcal{Q}_A$. All solutions $Q \in \mathcal{Q}_A$ appear in the asymptotic expansion, because they have different exponential terms $e^{V(Q)/h}$ by assumption. Since this includes all Galois conjugates of a solution $Q \in \mathcal{Q}_A$, this implies $[Q] = 0$ by Theorem 2.2.3. □

The proof of the theorem already gives hints as to why the modularity does not require all solutions of the Nahm equation to vanish in $\mathcal{B}(\mathbb{C})$. Two (or more) solutions $Q \in \mathcal{Q}_A$ might have the same volume $V(Q)$ and thus the same exponential term $e^{V(Q)/h}$. If, in addition to this, the asymptotic contributions associated to these solutions add up to zero, these solutions are not relevant for the asymptotics and the modularity. We discuss the phenomenon in Section 6.2 in more detail.

4.4. Proof of Theorem 4.3.1

In this section we prove Theorem 4.3.1. Therefore, we need some standard asymptotic results.

4.4.1. Asymptotics of $(wq;q)_\infty$

We prove an asymptotic expansion of the q-Pochhammer symbol. This is a refinement of Lemma 2.1 in [GZ21] following Lemma 4 in [Whe23].

Therefore, we need a version of the dilogarithm (Section 2.1) with rotated branch cuts. For $\varphi \in \mathbb{C}, |\varphi| = 1, |\arg\varphi| < \frac{\pi}{2}$ let $\widetilde{\log}$ be a version of the logarithm such that $\mathrm{Li}_1^\varphi(e^{-iv}) = -\widetilde{\log}(1-e^{-iv})$ has branch cuts whenever $\mathrm{Re}((v+2\pi n)/\varphi) = 0$ for some $n \in \mathbb{Z}$ and $\mathrm{Re}(v) > 0$. In other words, the principal branch cuts are rotated by φ, cf. Figure 4.2.

For $v \in \mathbb{C}$ not on a branch cut we define a version of the dilogarithm by

$$\mathrm{Li}_2^\varphi(e^{-iv}) \;=\; \int_0^\infty \mathrm{Li}_1^\varphi\left(e^{-\varphi x - iv}\right)dx, \tag{4.26}$$

where we avoid the branch cuts of $\mathrm{Li}_1^\varphi\left(1-e^{-\varphi x - iv}\right)$ in the contour, cf. Figure 4.2. Then $\mathrm{Li}_2^\varphi(e^{-iv})$ has the same branch cuts as $\mathrm{Li}_1^\varphi(e^{-iv})$ and jumps by $2\pi v$ when v crosses a branch cut, see Section 2.1.

As in subsection 2.1, the polylogarithm Li_m^φ for $m \in \mathbb{Z}_{\le 0}$ can de defined inductively by

$$\mathrm{Li}_m^\varphi(z) \;:=\; \frac{1}{z}\frac{d}{dz}\,\mathrm{Li}_{m+1}^\varphi(z), \tag{4.27}$$

where we note that Li_m^φ for $m \le 0$ is independent of φ and the branching of Li_1^φ. Therefore, Li_m^φ for $m < 0$ agrees with the polylogarithm Li_m defined in (2.1).

Lemma 4.4.1. *Let $w = e^{iv} \in \mathbb{C}$ such that if $\mathrm{Re}(v) > 0$ then $\mathrm{Re}((v+2\pi n)/\varphi) \neq 0$ for all $n \in \mathbb{Z}$. Moreover, let $\zeta \in \mathbb{C}$ be a root of unity of order $m \in \mathbb{N}$. Then we have as $h \to 0$ on a ray in the right half-plane, i.e., $q = \zeta e^{-h/m} \to \zeta$*

$$(wq;q)_\infty \;=\; \exp\left(-\frac{\mathrm{Li}_2^\varphi(w^m)}{mh} - \frac{1}{2}\,\mathrm{Li}_1^\varphi(w^m) + \sum_{t=1}^{m}\frac{t}{m}\,\mathrm{Li}_1^\varphi(\zeta^t w) + \psi_{w,\zeta}(h)\right), \tag{4.28}$$

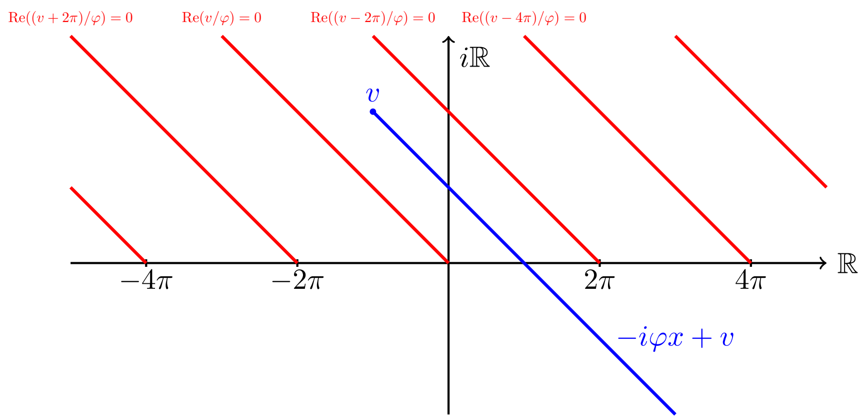


Figure 4.2.: The branch cuts of $\mathrm{Li}_s^{\varphi}(e^{-iv})$, $s = 1, 2$ and $-i\varphi x + v$ for $x \geq 0$

where $\psi_\zeta(h; w) \in \mathbb{C}[[h]]$ has an asymptotic expansion as $h \to 0$

$$\psi_\zeta(h; w) = -\sum_{s=2}^{N}\sum_{t=1}^{m} B_s\left(1 - \frac{t}{m}\right)\mathrm{Li}_{2-s}(\zeta^t w)\frac{h^{s-1}}{s!} + O(h^N) \tag{4.29}$$

for all $N > 0$.

Proof. Throughout the proof we write $h = \varphi|h|$ where $\varphi \in \mathbb{C}$ with $|\varphi| = 1$, $|\arg(\varphi)| < \frac{\pi}{2}$. We have

$$\widetilde{\log}(wq; q)_\infty = \sum_{n\geq 1}\widetilde{\log}(1 - wq^n) = \sum_{t=0}^{m-1}\sum_{k\geq 1}\widetilde{\log}\left(1 - \zeta^{-t}we^{-\varphi|h|(km-t)/m}\right) \tag{4.30}$$

and apply the Euler-Maclaurin summation formula [Zag06, p.13] to obtain for every $N \in \mathbb{N}$

$$\begin{aligned}\widetilde{\log}(wq; q)_\infty = &\sum_{t=0}^{m-1}\frac{1}{|h|}\int_0^\infty \widetilde{\log}\left(1 - \zeta^{-t}we^{-\varphi(x-t|h|/m)}\right)dx \\ &+ \sum_{n=0}^{N}\frac{(-1)^n B_{n+1}}{(n+1)!}\frac{d^n}{dx^n}\widetilde{\log}\left(1 - \zeta^{-t}we^{-\varphi(x-t|h|/m)}\right)dx|_{x=0}\,|h|^n + \mathcal{E}_{t,N}\end{aligned} \tag{4.31}$$

with

$$\mathcal{E}_{t,N} := |h|^N\int_0^\infty \mathrm{Li}_{1-N}\left(\zeta^{-t}we^{-\varphi(x-t|h|/m)}\right)\frac{\overline{B}_N(x)}{N!}dx. \tag{4.32}$$

The Euler-Maclaurin summation formula applies in this case, as the function defined by $x \mapsto \widetilde{\log}\left(1 - \zeta^{-t}we^{-\varphi(x-t|h|/m)}\right)$ and all of its derivatives

$$\frac{d^n}{dx^n}\widetilde{\log}\left(1 - \zeta^{-t}we^{-\varphi(x-t|h|/m)}\right) = -(-1)^n\varphi^n\,\mathrm{Li}_{1-n}^{\varphi}(\zeta^{-t}we^{-\varphi x+ht/m}) \tag{4.33}$$

are of rapid decay as $x \to \infty$.

We have

$$\begin{aligned}\frac{1}{|h|}\int_0^\infty \widetilde{\log}\left(1 - \zeta^{-t}we^{-\varphi(x-t|h|/m)}\right)dx &= -\frac{1}{z}\mathrm{Li}_2^{\varphi}(e^{ht/m}\zeta^{-t}w) \\ &= -\frac{1}{z}\sum_{l\geq 0}\frac{\mathrm{Li}_{2-l}^{\varphi}(\zeta^{-t}w)}{l!}\left(\frac{ht}{m}\right)^l\end{aligned} \tag{4.34}$$

where $\widetilde{\mathrm{Li}}_s^{\varphi}$ are the polylogarithms defined in (4.27). Evaluating the derivatives (4.33) at $x=0$ gives

$$\begin{aligned}\frac{d^n}{dx^n}\widetilde{\log}\left(1-\zeta^{-t}we^{-\varphi(x-t|h|/m)}\right)|_{x=0} &= -(-1)^n\varphi^n\,\mathrm{Li}_{1-n}^{\varphi}(\zeta^{-t}we^{ht/m}) \\ &= -(-1)^n\varphi^n\sum_{k\geq 0}\frac{\mathrm{Li}_{1-n-k}^{\varphi}(\zeta^{-t}w)}{k!}\left(\frac{ht}{m}\right)^k.\end{aligned} \tag{4.35}$$

Hence, (4.31) can be written as

$$\begin{aligned}\log(wq;q)_\infty &= -\sum_{t=0}^{m-1}\frac{1}{h}\sum_{l\geq 0}\frac{\mathrm{Li}_{2-l}^{\varphi}(\zeta^{-t}w)}{l!}\left(\frac{ht}{m}\right)^m \\ &\quad -\sum_{t=0}^{m-1}\sum_{n=0}^{N}\frac{B_{n+1}}{(n+1)!}\sum_{k\geq 0}\frac{\mathrm{Li}_{1-n-k}^{\varphi}(\zeta^{-t}w)}{k!}\left(\frac{t}{m}\right)^k h^{k+n}+\mathcal{E}_{t,N}\end{aligned} \tag{4.36}$$

by using the distribution property (2.11) of the dilogarithm. Moreover, shifting $n\mapsto n-1$ and summing over $s=n+k=1,\ldots,N+1$ shows that $\log(wq;q)_\infty$ is equal to

$$\begin{aligned}&-\frac{\mathrm{Li}_2^{\varphi}(w^m)}{mh}-\sum_{t=0}^{m-1}\sum_{s=1}^{N+1}\sum_{n=0}^{s}\binom{s}{n}B_n\left(\frac{t}{m}\right)^{s-n}\mathrm{Li}_{2-s}^{\varphi}(\zeta^{-t}w)\frac{h^{s-1}}{s!}+\mathcal{E}_{t,N}+O(h^N) \\ =&-\frac{\mathrm{Li}_2^{\varphi}(w^m)}{mh}-\sum_{t=0}^{m-1}\mathrm{Li}_1^{\varphi}(\zeta^{-t}w)\left(\frac{t}{m}-\frac{1}{2}\right)-\sum_{t=0}^{m-1}\sum_{s\geq 2}B_s\left(\frac{t}{m}\right)\mathrm{Li}_{2-s}(\zeta^{-t}w)\frac{h^{s-1}}{s!} \\ &+\mathcal{E}_{t,N}+O(h^N),\end{aligned} \tag{4.37}$$

where we collect all terms with $s\geq N$ in $O(h^N)$. Replacing t by $m-t\in\{1,\ldots,m\}$ yields

$$\begin{aligned}&-\frac{\mathrm{Li}_2^{\varphi}(w^m)}{mh}+\sum_{t=1}^{m}\mathrm{Li}_1^{\varphi}(\zeta^{t}w)\left(\frac{t}{m}-\frac{1}{2}\right)-\sum_{t=0}^{m-1}\sum_{s\geq 2}B_s\left(1-\frac{t}{m}\right)\mathrm{Li}_{2-s}(\zeta^{t}w)\frac{h^{s-1}}{s!}+\mathcal{E}_{t,N} \\ =&-\frac{\mathrm{Li}_2^{\varphi}(w^m)}{mh}-\frac{1}{2}\mathrm{Li}_1^{\varphi}(w^m)+\sum_{t=0}^{m-1}\frac{t}{m}\mathrm{Li}_1^{\varphi}(\zeta^{t}w)+\psi_{w,\zeta}(h)+O(h^N),\end{aligned} \tag{4.38}$$

where we used

$$\sum_{t=1}^{m}\mathrm{Li}_1^{\varphi}(\zeta^t w) = \sum_{t=1}^{m}\widetilde{\log}(1-\zeta^t w) = \widetilde{\log}\left(\prod_{t=1}^{m}1-\zeta^t w\right) = \widetilde{\log}(1-w^m) = \mathrm{Li}_1^{\varphi}(w^m). \tag{4.39}$$

Note that $\mathrm{Li}_{1-N}(z)\in(1-z)^N\mathbb{C}[z]$ for $N>0$. Our assumption implies that $we^{-\varphi x|h|}\neq 1$. There exists $C>0$ such that

$$|\,\mathrm{Li}_{1-N}(we^{-\varphi(|h|x+t|h|/m)})|<C|we^{-\varphi(|h|x+t|h|/m)}|<C|we^{-\varphi x|h|)}|.$$

We obtain for some $D > 0$

$$
\begin{aligned}
|\mathcal{E}_{N,t}| &\leq \frac{|h|^N}{N!} C \int_0^\infty |we^{-\varphi x|h|}|\overline{B}_N(x)dx \\
&= \frac{|h|^N}{N!} C \sum_{x_0=0}^{\infty} \int_0^\infty |we^{-\varphi|h|(x_0+x)}|B_N(x)dx \\
&= \frac{|h|^N}{N!} C|w| \sum_{x_0=0}^{\infty} |e^{-\varphi|h|x_0}| \int_0^1 |e^{-\varphi|h|x}|B_N(x)dx \\
&= \frac{|h|^N}{N!} \frac{C|w|}{1-|e^{-\varphi|h|}|} D \; = \; O(|h|^N),
\end{aligned}
\tag{4.40}
$$

since $|e^{-\varphi|h|}| < 1$. In particular, $\psi_{w,\zeta}(h)$ has the claimed asymptotic expansion. This completes the proof. ☐

4.4.2. Miscellaneous asymptotic results

Moreover, we require some basic asymptotic estimates. Therefore, recall the convention $e(x) = e^{2\pi i x}$ for $x \in \mathbb{C}$.

Lemma 4.4.2. *The following are true.*

1. *Let $\alpha \in \mathbb{R}$. As $|t| \to \infty$ for t on a ray in $\mathbb{C}$ we have with $\pm = \operatorname{sign}(\operatorname{Re}(t))$*

$$
\sin\left(\alpha(it - n_0)\right) \; = \; \pm\frac{1}{2i}\exp\left(\pm\alpha(t - n_0 i)\right)\left(1 + o(t^{-L})\right) \tag{4.41}
$$

for all $L > 0$.

2. *Let $\zeta = e(\frac{r}{m})$ be a root of unity of order $m \in \mathbb{N}$. As $h \to 0$ on a ray in the right half-plane, i.e., $q = \zeta e^{-h} \to \zeta$, we have*

$$
(q;q)_\infty \; = \; \exp\left(-\frac{\pi^2}{6m^2h} + \frac{h}{24}\right)\sqrt{\frac{2\pi}{mh}}\, e\left(\frac{s(-r,m)}{2}\right)\left(1 + o(h^L)\right) \tag{4.42}
$$

for all $L > 0$. Here, $s(r,m)$ is the Dedekind sum defined in (2.57).

3. *Assuming the notation above, if m is even, we have*

$$
(-q;q)_\infty \; = \; e^{-\frac{\pi^2}{6m^2h} + \frac{h}{12}} Q(\zeta)\left(1 + o(h^L)\right) \tag{4.43}
$$

as $h \to 0$ in the right half-plane for all $L > 0$, where

$$
Q(\zeta) \; = \; e\left(\frac{s(-r,\frac{m}{2}) - s(-r,m)}{2}\right). \tag{4.44}
$$

Proof. We prove each part of the Lemma separately as follows.

1. We have

$$
\begin{aligned}
\sin(\alpha(it - n_0)) &= \frac{e^{-\alpha(t+in_0)} - e^{\alpha(t+in_0)}}{2i} \\
&= \begin{cases} \frac{1}{2i}\exp\left(\alpha(t+in_0)\right)\left(1 - e^{-\alpha(2t+2in_0)}\right), & \text{if } \operatorname{Re}(t) > 0, \\ -\frac{1}{2i}\exp\left(-\alpha(t+in_0)\right)\left(1 - e^{\alpha(2t+2in_0)}\right), & \text{if } \operatorname{Re}(t) < 0. \end{cases}
\end{aligned}
\tag{4.45}
$$

In each case, the second exponential becomes exponentially small as $t \to \infty$.

2. From (2.53), we know that the eta function
$$\eta(\tau) \;=\; q^{\frac{1}{24}}(q;q)_\infty, \tag{4.46}$$
where $q=e^{2\pi i\tau}$ and $\tau\in\mathbb{H}$ satisfies the modular transformation formula
$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) \;=\; e\left(\frac{a+d}{24c}-\frac{s(d,c)}{2}-\frac{1}{8}\right)(c\tau+d)^{\frac{1}{2}}\eta(\tau) \tag{4.47}$$
for all $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})$ with $c>0$. Hence, for $\frac{r}{m}\in\mathbb{Q}$ with $(r,m)=1$ we choose $a,b\in\mathbb{Z}$ such that $\left(\begin{smallmatrix} -a & -b\\ m & -r\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})$. Then we obtain for all $\tau\in\mathbb{H}$ with $q=e^{2\pi i\tau}$
$$(q;q)_\infty \;=\; q^{-\frac{1}{24}}\eta(\tau) \;= q^{-\frac{1}{24}}(m\tau-r)^{-\frac{1}{2}}e\left(\frac{a+r}{24m}+\frac{s(-r,m)}{2}+\frac{1}{8}\right)\eta\left(\frac{a\tau+b}{-m\tau+r}\right). \tag{4.48}$$
Setting $\tau=\frac{r}{m}-\frac{h}{2\pi i}$, we have $m\tau-r=-\frac{mh}{2\pi i}$ and
$$\frac{a\tau+b}{-m\tau+r} \;=\; 2\pi i\frac{ar+bm}{m^2h}-\frac{a}{m} \;=\; \frac{2\pi i}{m^2h}-\frac{a}{m}. \tag{4.49}$$
Because h is in the right half-plane, we have $\tau\in\mathbb{H}$ and we obtain
$$\begin{aligned}\eta\left(\frac{a\tau+b}{-m\tau+r}\right) \;&=\; \eta\left(\frac{2\pi i}{m^2h}-\frac{a}{m}\right)\\ &= e\left(\frac{1}{24}\left(\frac{2\pi i}{m^2h}-\frac{a}{m}\right)\right)\prod_{n\geq 1}\left(1-e\left(n\left(\frac{2\pi i}{m^2h}-\frac{a}{m}\right)\right)\right)\\ &= \exp\left(-\frac{\pi^2}{6m^2h}\right)e\left(\frac{-a}{24m}\right)\prod_{n\geq 1}\left(1-\exp\left(\frac{-4\pi^2 n}{m^2h}-\frac{2\pi i a n}{m}\right)\right).\end{aligned} \tag{4.50}$$
As $h\to 0$ on a ray in the right half-plane we have
$$\prod_{n\geq 1}\left(1-\exp\left(\frac{-4\pi^2 n}{m^2h}-\frac{2\pi i a n}{m}\right)\right) \;=\; (1+o(h^L)) \tag{4.51}$$
for all $L>0$. Hence, we obtain with $q=e^{2\pi i\tau}=e^{2\pi i\frac{r}{m}}e^{-h}$
$$(q;q)_\infty \;=\; \exp\left(-\frac{\pi^2}{6m^2h}+\frac{h}{24}\right)\sqrt{\frac{2\pi}{mh}}e\left(\frac{s(-r,m)}{2}\right)(1+o(h^L)) \tag{4.52}$$
as claimed.

3. The claim follows from the previous statement and the identity $(-q;q)_\infty=\frac{(q^2;q^2)_\infty}{(q;q)_\infty}$.

$\square$

4.4.3. Proof of Theorem 4.3.1

Proof of Theorem 4.3.1. Consider
$$\begin{aligned} f_{A,b}(q) \;&=\; \sum_{n\in\mathbb{Z}^r}\frac{q^{\frac{1}{2}n^TAn+b^Tn}}{(q)_{n_1}\cdots(q)_{n_r}}\\ &= (q;q)_\infty^{-r}\sum_{n\in\mathbb{Z}^r}q^{\frac{1}{2}n^TAn+b^Tn}\prod_{j=1}^r(q^{n_j}q;q)_\infty\end{aligned} \tag{4.53}$$

Following the methods from [Zag07], we can use the Euler-Maclaurin formula to replace the sum in (4.53) by an integral, without changing the asymptotic behaviour of the Nahm sum. Therefore, we have that $f_{A,b}(e^{-h})$ as $h \to 0$ is asymptotically equal to

$$\begin{aligned}
&(q;q)_\infty^{-r} \int_{\mathbb{R}^r} q^{\frac{1}{2}n^TAn+b^Tn} \prod_{j=1}^{r}(q^{n_j}q;q)_\infty dn \\
&= (q;q)_\infty^{-r} \int_{\mathbb{R}^r} e^{-h\frac{1}{2}n^TAn-hb^Tn} \prod_{j=1}^{r}(e^{-hn_j}q;q)_\infty dn \qquad n=\frac{t}{h} \\
&= (h(q;q)_\infty)^{-r} \int_{\mathbb{R}^r} e^{-\frac{1}{2h}t^TAt-b^Tt} \prod_{j=1}^{r}(e^{-t}q;q)_\infty dt.
\end{aligned} \tag{4.54}$$

As $q = e^{-h} \to 1$, i.e., $h \to 0$ we write using Lemma 4.4.1 and Lemma 4.4.2

$$f_{A,b}(e^{-h}) = \left(\frac{e^{\pi^2/6h-h/24}}{\sqrt{2\pi h}}\right)^r \int_{\mathbb{R}^r} \exp\Big(\frac{g(t)}{h}\Big) R(t;h)dt \tag{4.55}$$

where

$$\begin{aligned}
g(t) &\coloneqq -\frac{1}{2}t^TAt - \sum_{j=1}^{r} \mathrm{Li}_2(e^{-t_j}) \\
R(t;h) &\coloneqq \exp\Big[-b^Tt + \sum_{j=1}^{r}\frac{1}{2}\mathrm{Li}_1(e^{-t_j}) + \psi_1(e^{-t_j};h)\Big]
\end{aligned} \tag{4.56}$$

with ψ_1 defined in Lemma 4.4.1. For the derivatives of g we have

$$\begin{aligned}
g'(t) &= -At - (\log(1-e^{-t_1}), \dots, \log(1-e^{-t_r}))^T, \\
g''(t) &= A + \mathrm{diag}(\mathrm{Li}_0(e^{-t_j}))_j) = -\widetilde{A}(Q), \\
g^{(k)}(t) &= \mathrm{Li}_{2-k}(e^{-t_j}) \qquad \text{for } k \geq 3,
\end{aligned} \tag{4.57}$$

cf. (4.11). Hence, the stationary points $t_0 \in \mathbb{C}^r$ of g are given by

$$g'(t_0) = 0 \quad \Leftrightarrow \quad \log(1-e^{-t_i}) = -\sum_{j=1}^{r} A_{i,j}t_{0,j}, \quad i = 1, \dots r. \tag{4.58}$$

Exponentiating yields the Nahm equations

$$1 - Q_i = \prod_{j=1}^{r} Q_j^{A_{i,j}}, \quad Q_i = e^{-t_{0,i}} \tag{4.59}$$

for $i = 1, \cdots, r$, as in (3.8). The maximal exponential contribution is then given by $t_{0,i} = -\log(Q_i)$, where Q maximises $\mathrm{Re}(V(Q)/h)$ and we choose the principal branch of the logarithm. We expand around $t \mapsto t_0 + t$ for $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ and note that the contribution for large t can be neglected. Then we obtain

$$\begin{aligned}
f_{A,b}(q) = &\left(\frac{e^{\pi^2/6h-h/24}}{\sqrt{2\pi h}}\right)^r \exp\Big(\frac{g(t_0)}{z}\Big)\exp(-b^Tt_0) \\
&\int_{(-\varepsilon,\varepsilon)^r} \exp\Big[-\frac{t^T\widetilde{A}t}{2z} - b^Tt - \sum_{j=1}^{r}\sum_{k\geq 3}\mathrm{Li}_{2-k}(Q_j)\frac{(-t)^k}{zk!} + \frac{1}{2}\mathrm{Li}_1(Q_je^{-t_j}) \\
&\qquad - \sum_{s\geq 2} B_s \,\mathrm{Li}_{2-s}(Q_je^{-t_j})\frac{z^{s-1}}{s!}\Big]dt.
\end{aligned} \tag{4.60}$$

Expanding the polylogarithms using (2.1) and substituting $t \mapsto th^{1/2}$ gives

$$\begin{aligned} f_{A,b}(q) \;=\; & \left(\frac{e^{\pi^2/6h - h/24}}{\sqrt{2\pi}}\right)^r \exp\Big(\frac{g(t_0)}{h}\Big) \exp\Big(-b^T t_0 - \frac{1}{2}\log(1-Q)\Big) \\ & \int_{(-\varepsilon,\varepsilon)^r} \exp\Big[-\frac{t^T \widetilde{A} t}{2} - b^T t h^{1/2} - \sum_{j=1}^{r} \sum_{k\geq 3} \mathrm{Li}_{2-k}(Q_j) \frac{(-t)^k h^{k/2}}{hk!} \\ & + \sum_{l\geq 1} \frac{1}{2} \mathrm{Li}_{2-1-l}(Q_j) \frac{(-t_j)^l h^{l/2}}{l!} - \sum_{s\geq 2} \sum_{\nu \geq 0} B_s \, \mathrm{Li}_{2-s-\nu}(Q_j) \frac{h^{s+\nu/2-1}(-t_j)^\nu}{s!\nu!}\Big] dt. \end{aligned} \tag{4.61}$$

Reorganising the summands yields

$$\begin{aligned} f_{A,b}(q) \;=\; & \left(\frac{e^{\pi^2/6h - h/24}}{\sqrt{2\pi}}\right)^r \exp\left(\frac{g(t_0)}{h}\right) \exp\Big(-b^T t_0 - \frac{1}{2}\log(1-Q) - \frac{h}{12}\sum_{j=1}^{r} \frac{Q_j}{1-Q_j}\Big) \\ & \int_{(-\varepsilon,\varepsilon)^r} \exp\Big[-\frac{t^T \widetilde{A} t}{2} - \sum_{j=1}^{r} \Big(b_j + \frac{1}{2}\mathrm{Li}_0(Q_j)\Big) t h^{1/2} \\ & - \sum_{j=1}^{r} \sum_{\substack{s\geq 2, \nu\geq 0: \\ p=s+\nu\geq 3}} B_s \, \mathrm{Li}_{2-s-\nu}(Q_j) \frac{h^{s+\nu/2-1}(-t_j)^\nu}{s!\nu!}\Big] dt, \end{aligned} \tag{4.62}$$

which can be written as

$$\begin{aligned} & (2\pi)^{-\frac{r}{2}} \exp\left(\frac{r\frac{\pi^2}{6} + g(t_0)}{h}\right) \exp\left(-b^T t_0 - \frac{1}{2}\log(1-Q) - \frac{h}{24}\sum_{j=1}^{r} \frac{1+Q_j}{1-Q_j}\right) \\ & \int_{(-\varepsilon,\varepsilon)^r} \exp\Big[-\frac{t^T \widetilde{A} t}{2} - \sum_{j=1}^{r} \left(b_j + \frac{1}{2}\mathrm{Li}_0(Q_j)\right) t h^{1/2} \\ & - \sum_{j=1}^{r} \sum_{p\geq 3} B_s \left(-\frac{t}{h^{1/2}}\right) \mathrm{Li}_{2-s-\nu}(Q_j) \frac{h^{p-1}}{p!}\Big] dt. \end{aligned} \tag{4.63}$$

We note that

$$r\frac{\pi^2}{6} + g(t_0) \;=\; r\frac{\pi^2}{6} + \sum_{j=1}^{r} \mathrm{Li}_2(Q_j) + \log(Q_j)\log(1-Q_j) \;=\; V(Q), \tag{4.64}$$

where $V(Q)$ is the volume defined after Proposition 2.2.7 and

$$\exp\Big(-b^T t_0 - \frac{1}{2}\sum_{j=1}^{r} \log(1-Q_r)\Big) \;=\; \prod_{j=1}^{r} \frac{Q_j^{b_j}}{\sqrt{1-Q_j}}. \tag{4.65}$$

Moreover, because $\varepsilon > 0$ is arbitrary small, the integral can be written as a formal Gaussian integral as defined in (4.4). This establishes

$$f_{A,b}(e^{-h}) \;=\; e^{V(Q)/z}\, \Phi^{(Q)}_{A,b}(h)\,(1+O(z^L)) \;=\; \widehat{\Phi}^{(Q)}_{A,b}(h)\,(1+O(z^L)) \tag{4.66}$$

for all $L > 0$ and completes the proof. $\square$

5. Vector-valued modularity

In this Chapter, we discuss "Nahm's conjecture", assuming that the images of all $Q \in \mathcal{Q}_A$ vanish in $\mathcal{B}(\mathbb{C})$. As discussed in Section 3.3, this is believed to imply the modularity of $f_{A,b,c}(q)$ for *some* $b \in \mathbb{Q}^r$, $c \in \mathbb{Q}$ and *some* congruence subgroup. We refine this expectation by giving predictions of which b's and c's have to be taken and how the vector of modular Nahm sums transforms under $\mathrm{SL}_2(\mathbb{Z})$. A similar approach was also proposed in [Nah23].

5.1. Representation for $\mathrm{SL}_2(\mathbb{Z})$

Assume that $(f_1(q), \cdots, f_n(q))$ is a vector-valued modular function for $\mathrm{SL}_2(\mathbb{Z})$. Then the transformation under $\tau \mapsto -\frac{1}{\tau}$ is given by

$$f_j(q) \;=\; \sum_{k=1}^{n} a_{j,k}\, f_k(\tilde{q}), \tag{5.1}$$

where $q = e^{2\pi i \tau}$ and $\tilde{q} = e^{-2\pi i/\tau}$, for some $a_{j,k} \in \mathbb{C}$. We can suppose that $f_k(\tilde{q}) \in \tilde{q}^{\alpha_k}(1+\tilde{q}\mathbb{Z}[[\tilde{q}]])$ for some $\alpha_k \in \mathbb{Q}$. Then each summand in (5.1) with $q = e^{-h}$, $\tilde{q} = e^{-4\pi^2/h}$ leads to a summand in the (formal) asymptotics

$$f_j(e^{-h}) \;=\; \sum_{k=0}^{n} a_{j,k}\, e^{-4\pi^2\alpha_k/h}\,\left(1 + o(e^{-4\pi^2\alpha_k/h})\right), \tag{5.2}$$

as $h \to 0$ on a ray in the right half-plane. The asymptotics of Nahm sums (Theorem 4.3.1)

$$f_{A,b,c}(e^{-h}) \;=\; \sum_{Q\in\mathcal{Q}_A} e^{V(Q)/h}\,\Phi^{(Q)}_{A,b,c}(h) \tag{5.3}$$

suggests that, if $f_{A,b,c}(q)$ is modular as part of a vector-valued modular function as above, the components $f_k(q)$ of the vector-valued modular function correspond to the solutions $Q \in \mathcal{Q}_A$ such that $\alpha_k = -\frac{V(Q)}{4\pi^2} \in \mathbb{Q}$. (Recall that $[Q] = 0 \in \mathcal{B}(\mathbb{C})$ implies that $V(Q) \in \pi^2\mathbb{Q}$ and indeed, this observation was the original motivation for Nahm's conjecture.)

Moreover, we expect that the coefficients $a_{j,k}$ in (5.1) are given by the power series $\Phi^{(Q)}_{A,b,c}(h)$ in (5.3) that are, in fact, constant in this case, i.e.,

$$\Phi^{(Q)}_{A,b,c}(h) \;=\; \Phi^{(Q)}_{A,b,c}(0) \;=\; \frac{\prod_{j=1}^{r} Q_j^{b_j}}{\sqrt{\delta_A(Q)}}, \tag{5.4}$$

where

$$\delta_A(Q) = \det\left(A + \mathrm{diag}\left(\frac{Q_i}{1-Q_i}\right)_{i=1,\dots,r}\right)\prod_{i=1}^{r}(1-Q_i), \qquad Q \in \mathcal{Q}_A. \tag{5.5}$$

Recall that $V(Q)$ denotes the principal branch of the volume and that $V(\log Q, \log(1-Q))$ is only defined modulo $4\pi^2$. In the proof of Theorem 4.3.1, we only took the leading contributions for each $Q \in \mathcal{Q}_A$ into consideration. The different shifts of $V(Q)$ by $4\pi^2$ lead to factors of $e^{-4\pi^2/h} = \tilde{q}$ in (5.3). Collecting all terms, we obtain a $\tilde{q}$-series attached to each contribution. If we (formally) keep track of the different branches, these $\tilde{q}$-series turn out to be Nahm sums for the same A and different b's.

5.1.1. Example: The Rogers-Ramanujan functions

We illustrate this behaviour using the Rogers-Ramanujan functions from Chapter 1, i.e., $A = 2$, and then summarise our observations and general expectations in subsection 5.1.2.

If we denote the solutions of (3.8), here $1 - Q = Q^2$, by $Q_1 = \frac{-1+\sqrt{5}}{2}$ and $Q_2 = \frac{-1-\sqrt{5}}{2}$, we have

$$V(Q_1) \;=\; \frac{1}{60}4\pi^2, \qquad V(Q_2) \;=\; -\frac{11}{60}4\pi^2.$$

By Theorem 4.3.1, as $h \to 0$

$$f_{2,0}(e^{-h}) \;=\; e^{V(Q_1)/h}\, \Phi_{2,0}^{(Q_1)}(h) \tag{5.6}$$

asymptotically to all orders, where the power series turns out to be

$$\Phi_{2,0}^{(Q_1)}(h) \;=\; \frac{1}{\sqrt{2-2Q_1+Q_1}} e^{-h/60} \;=\; \frac{2}{\sqrt{5}} \sin\left(\frac{2\pi}{5}\right)\, e^{-h/60}. \tag{5.7}$$

Hence,

$$f_{2,0,-\frac{1}{60}}(e^{-h}) \;=\; e^{V(Q_1)/h}\, \frac{2}{\sqrt{5}} \sin\left(\frac{2\pi}{5}\right) + O(h^N)$$

for all $N \geq 0$. We can compute the asymptotics of the difference between the left and the right hand side and obtain (numerically or using modularity)

$$f_{2,0,-\frac{1}{60}}(e^{-h}) - e^{V(Q_1)/h}\, \frac{2}{\sqrt{5}} \sin\left(\frac{2\pi}{5}\right) \;=\; e^{V(Q_2)/h}\, \Phi_{2,0,-\frac{1}{60}}^{(Q_2)}(h)$$

asymptotically, where

$$\Phi_{2,0,-\frac{1}{60}}^{(Q_2)}(h) \;=\; \frac{2}{\sqrt{5}} \sin\left(\frac{\pi}{5}\right). \tag{5.8}$$

The asymptotics of the difference in this equation is given by

$$\begin{aligned} &f_{2,0,-\frac{1}{60}}(e^{-h}) - e^{V(Q_1)/h}\, \frac{2}{\sqrt{5}} \sin\left(\frac{2\pi}{5}\right) - e^{V(Q_2)/h}\, \frac{2}{\sqrt{5}} \sin\left(\frac{\pi}{5}\right) \\ &= e^{(V(Q_1)+4\pi^2)/h}\, \frac{2}{\sqrt{5}} \sin\left(\frac{2\pi}{5}\right). \end{aligned}$$

By repeating this procedure we find the expansion

$$\begin{aligned} f_{2,0,-\frac{1}{60}}(e^{-h}) \;&= e^{V(Q_1)/h}\, \frac{2}{\sqrt{5}} \sin\left(\frac{2\pi}{5}\right)\, (1 + \tilde{q} + \tilde{q}^2 + \tilde{q}^3 + 2\tilde{q}^4 + 2\tilde{q}^5 + 3\tilde{q}^6 + \cdots) \\ &\quad + e^{V(Q_2)/h}\, \frac{2}{\sqrt{5}} \sin\left(\frac{\pi}{5}\right)\, (1 + \tilde{q} + \tilde{q}^2 + \tilde{q}^3 + \tilde{q}^4 + \tilde{q}^5 + 2\tilde{q}^6 + \cdots), \end{aligned}$$

where $\tilde{q} = e^{-4\pi^2/h}$. This is exactly the modular transformation from equation (3.5)

$$f_{2,0,-\frac{1}{60}}(q) \;=\; \frac{2}{\sqrt{5}} \left(\sin\left(\frac{2\pi}{5}\right) f_{2,0,-\frac{1}{60}}(\tilde{q}) \;+\; \sin\left(\frac{\pi}{5}\right) f_{2,1,\frac{11}{60}}(\tilde{q}) \right).$$

Similarly, the asymptotics for $f_{2,1,\frac{11}{60}}$ gives rise to its modular transformation

$$f_{2,1,\frac{11}{60}}(q) \;=\; \frac{2}{\sqrt{5}}\left(\sin\left(\frac{\pi}{5}\right) f_{2,0,-\frac{1}{60}}(\tilde{q}) \;-\; \sin\left(\frac{\pi}{5}\right) f_{2,1,\frac{11}{60}}(\tilde{q})\right).$$

5.1.2. The components of the vector-valued modular function

Throughout we assume that $A \in \mathbb{Z}^{r\times r}$ is an even, symmetric and positive definite matrix. Moreover, we assume that $[Q]$ vanishes in the Bloch group for all solutions $Q \in \mathcal{Q}_A$. Then it is believed that $f_{A,b,c}(q)$ is a modular function for some $b \in \mathbb{Q}^r$, $c \in \mathbb{Q}$. We make the following predictions.

1. For every solution $Q \in \mathcal{Q}_A$ let v be the principal branch of $\log(1-Q)$. Then there exists a choice of $u = \log(Q)$ such that with

$$\begin{aligned} b(Q) \;&\coloneqq (Au - v)/2\pi i \in \mathbb{Z}^r, \\ c(Q) \;&\coloneqq\; -V(u,v)/4\pi^2 \in \mathbb{Q}, \end{aligned} \tag{5.9}$$

   the function $f_{A,b(Q),c(Q)}(q)$ is a modular function.

2. For all $Q, Q' \in \mathcal{Q}_A$, the power series $\Phi^{(Q')}_{A,b(Q),c(Q)}(h)$ are constant

$$\Phi^{(Q')}_{A,b(Q),c(Q)}(h) \;=\; \Phi^{(Q')}_{A,b(Q),c(Q)}(0) \;=\; \frac{\prod_{j=1}^{r} {Q'}_j^{b(Q)_j}}{\sqrt{\delta_A(Q')}}, \tag{5.10}$$

   with $\delta_A(Q')$ defined in (5.5).

3. Define the vector-valued function

$$F_A(q) \;=\; \big(f_{A,b(Q),c(Q)}(q)\big)_{Q\in\mathcal{Q}_A}. \tag{5.11}$$

   Then the vector-valued function $\tilde{F}_A(\tau) \;=\; F_A(e^{2\pi i\tau})$ with components

$$\tilde{f}_{A,b(Q),c(Q)}(\tau) \;=\; f_{A,b(Q),c(Q)}(e^{2\pi i\tau}), \qquad \tau \in \mathbb{H}, \tag{5.12}$$

   is a vector-valued modular function for $\mathrm{SL}_2(\mathbb{Z})$. The transformation under $\tau \mapsto \tau+1$ is (trivially) given by

$$\tilde{f}_{A,b(Q),c(Q)}(\tau+1) \;= e^{2\pi i c(Q)}\,\tilde{f}_{A,b(Q),c(Q)}(\tau), \tag{5.13}$$

   since $f_{A,b(Q),c(Q)}(q) \in q^{c(Q)}\mathbb{Z}[[q]]$, and the transformation under $\tau \mapsto -\frac{1}{\tau}$ is by

$$\tilde{f}_{A,b(Q),c(Q)}\Big(-\frac{1}{\tau}\Big) \;=\; \sum_{Q'\in\mathcal{Q}_A} \frac{Q'^{b(Q)}}{\sqrt{\delta_A(Q')}} \tilde{f}_{A,b(Q'),c(Q')}(\tau), \tag{5.14}$$

   for a square-root of $\delta_A(Q') \in \mathbb{C}$, defined in (5.5).

In particular, the map induced by (5.12) and (5.14) for the generators $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ is a representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}^{|\mathcal{Q}_A|}$.

We will see below that sometimes the set $\mathcal{Q}_A \subset \mathbb{C}$ is not enough, and we also have to include formal solutions of the Nahm equation containing $\pm\infty$.

5.1.3. More examples

Example 5.1.1. We illustrate the idea from subsection 5.1.2 using the matrix $A = \left(\begin{smallmatrix} 4 & 2 \\ 2 & 2 \end{smallmatrix}\right)$. The Nahm equation (3.8) for A can be written as

$$1 - Q_1 \;=\; Q_1^4 Q_2^2, \qquad 1 - Q_2 \;=\; Q_1^2 Q_2^2 \tag{5.15}$$

and has explicit solutions

$$Q^{(k)} \;:=\; \left(1 - \frac{\sin^2((2k-1)\frac{\pi}{7})}{\sin^2((4-l)(2k-1)\frac{\pi}{7})}\right)_{l=1,2} \tag{5.16}$$

for $k = 1,2,3$. Sine all solutions are totally real, their images vanish in $\mathcal{B}(\mathbb{C})$ (Theorem 2.2.3) and thus we expect that $f_{A,b,c}(q)$ is a modular function for some $b \in \mathbb{Q}^2, c \in \mathbb{Q}$. Indeed, according to the Andrews-Gordon identities [And74], we have

$$\begin{aligned}
f_{A,\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)}(q) \;&=\; \prod_{\substack{n\geq 1 \\ n=0,\pm 3 \bmod 7}} (1-q^n) \;=\; \frac{1}{(q)_\infty} \sum_{n\in\mathbb{Z}} q^{7n^2/2+n/2}, \\
f_{A,\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}(q) \;&=\; \prod_{\substack{n\geq 1 \\ n=0,\pm 2 \bmod 7}} (1-q^n) \;=\; \frac{1}{(q)_\infty} \sum_{n\in\mathbb{Z}} q^{7n^2/2+3n/2}, \\
f_{A,\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)}(q) \;&=\; \prod_{\substack{n\geq 1 \\ n=0,\pm 1 \bmod 7}} (1-q^n) \;=\; \frac{1}{(q)_\infty} \sum_{n\in\mathbb{Z}} q^{7n^2/2+5n/2}.
\end{aligned} \tag{5.17}$$

As in Example 2.4.6, one can show that the function

$$\tilde{F}_A(\tau) \;:=\; F_A(q) \;:=\; \begin{pmatrix} f_{A,\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right),-\frac{1}{42}}(q) \\ f_{A,\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right),\frac{5}{42}}(q) \\ f_{A,\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right),\frac{17}{42}}(q) \end{pmatrix}, \qquad q = e^{2\pi i\tau} \tag{5.18}$$

is a vector-valued modular function fulfilling the transformations

$$\tilde{F}_A(\tau+1) \;=\; \begin{pmatrix} \zeta_{42}^{-1} & 0 & 0 \\ 0 & \zeta_{42}^{5} & 0 \\ 0 & 0 & \zeta_{42}^{17} \end{pmatrix} \tilde{F}_A(\tau), \qquad \zeta_{42} = e^{\pi i/21}, \tag{5.19}$$

$$\tilde{F}_A\left(\frac{-1}{\tau}\right) \;=\; \frac{2}{\sqrt{7}} \begin{pmatrix} \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{14}) & \cos(\frac{5\pi}{14}) \\ \cos(\frac{3\pi}{14}) & -\cos(\frac{5\pi}{14}) & -\cos(\frac{\pi}{14}) \\ \cos(\frac{5\pi}{14}) & -\cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{14}) \end{pmatrix} \tilde{F}_A(\tau). \tag{5.20}$$

We will construct the components of F_A as well as the modular transformation by means of subsection 5.1.2. With the logarithms

k	$Q^{(k)}$	$\log(Q^{(k)})$	$\log(1-Q^{(k)})$
1	$\begin{pmatrix} 0.8019377\cdots \\ 0.6920215\cdots \end{pmatrix}$	$\begin{pmatrix} -0.2207243\cdots \\ -0.3681383\cdots \end{pmatrix}$	$\begin{pmatrix} -1.619174\cdots \\ -1.177725\cdots \end{pmatrix}$
2	$\begin{pmatrix} -0.5549581\cdots \\ -4.048917\cdots \end{pmatrix}$	$\begin{pmatrix} -0.5888626\cdots + \pi i \\ 1.398450\cdots - \pi i \end{pmatrix}$	$\begin{pmatrix} -1.619174\cdots \\ -1.177725\cdots \end{pmatrix}$
3	$\begin{pmatrix} -2.246980\cdots \\ 0.3568959\cdots \end{pmatrix}$	$\begin{pmatrix} 0.8095869\cdots + \pi i \\ -1.030311\cdots \end{pmatrix}$	$\begin{pmatrix} -1.619174\cdots \\ -1.177725\cdots \end{pmatrix}$

we compute, following (5.9),

$$\begin{aligned} b(Q^{(1)}) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & c(Q^{(1)}) &= -\frac{1}{42}, \\ b(Q^{(2)}) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & c(Q^{(2)}) &= \frac{5}{42}, \\ b(Q^{(3)}) &= \begin{pmatrix} 2 \\ 1 \end{pmatrix}, & c(Q^{(3)}) &= \frac{17}{42}. \end{aligned} \tag{5.21}$$

Indeed, the computed values for $b(Q)$ and $c(Q)$ match with the values for b, c such that $f_{A,b,c}(q)$ is modular. This provides the 1:1 correspondence between the solutions $Q \in \mathcal{Q}$ and the components of $F_A(q)$.

The transformation for $\tau \mapsto \tau + 1$ in (5.19) follows from the fact that

$$f_{\left(\begin{smallmatrix} 4 & 2 \\ 2 & 2 \end{smallmatrix}\right),b,c}(q) \in q^c(1 + \mathbb{Z}[[q]])$$

for $b \in \mathbb{Z}$. In view of (5.14), we compute the coefficients

$$\Phi_{A,b}^{(Q)}(0) = \frac{\prod_{j=1}^{r} Q_j^{b_j}}{\sqrt{\delta_A(Q)}}, \tag{5.22}$$

where $\delta_A(Q)$ is defined in (5.5)

b	$\Phi_{A,b}^{(Q^{(1)})}(0)$	$\Phi_{A,b}^{(Q^{(2)})}(0)$	$\Phi_{A,b}^{(Q^{(3)})}(0)$
$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\frac{2}{\sqrt{7}}\cos(\frac{\pi}{14})$	$\frac{2}{\sqrt{7}}\cos(\frac{3\pi}{14})$	$\frac{2}{\sqrt{7}}\cos(\frac{5\pi}{14})$
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\frac{2}{\sqrt{7}}\cos(\frac{3\pi}{14})$	$-\frac{2}{\sqrt{7}}\cos(\frac{5\pi}{14})$	$-\frac{2}{\sqrt{7}}\cos(\frac{\pi}{14})$
$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$	$\frac{2}{\sqrt{7}}\cos(\frac{5\pi}{14})$	$-\frac{2}{\sqrt{7}}\cos(\frac{\pi}{14})$	$\frac{2}{\sqrt{7}}\cos(\frac{3\pi}{14})$

We note that the values agree with the coefficients in the matrix in (5.20).

Next, we consider the example $A = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ to illustrate two phenomena:

1. The Nahm equation (3.8) for A has solutions in $\mathbb{C}^2$, as well as formal solutions with entries $\pm\infty$. Hence, the ideas from subsection 5.1.2 are not directly applicable. Nevertheless, we will see that the philosophy still works and the formulas there will lead to the right result.

2. We will construct a vector-valued modular function following subsection 5.1.2 that includes all $f_{A,b,c}(q)$ with integral b. However, this will not cover all modular functions $f_{A,b,c}(q)$ with $b \in \mathbb{Q}^r$. Instead, the Nahm sums $f_{A,b,c}(q)$ with $b \in \mathbb{Q}^r$ will form a vector-valued modular function together with Nahm sums including $(-1)^{b^t n}$ in the summation. We discuss this behaviour in the context of non-even matrices in subsection 5.1.4.

Example 5.1.2. Let $A = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. From [VZ11, Table 2], it is known that $f_{A,b,c}$ is modular for the following values of $b \in \mathbb{Q}^2, c \in \mathbb{Q}$

b	c		$f_{A,b,c}(q)$
$\begin{pmatrix} \beta \\ -\beta \end{pmatrix}$	$\frac{\beta^2}{4} - \frac{1}{24}$	$\beta \in \mathbb{Q}$	$\frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z} + \frac{\beta}{2}} q^{n^2}$
$\frac{1}{2}\begin{pmatrix} -1 \\ -1 \end{pmatrix}$	$\frac{1}{48}$		$\frac{2}{\eta(\tau)} \sum_{n \in \mathbb{Z} + \frac{1}{4}} q^{n^2}$
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\frac{5}{24}$		$\frac{1}{2\eta(\tau)} \sum_{n \in \mathbb{Z} + \frac{1}{2}} q^{n^2}$

We have the obvious relations

$$\begin{aligned} 2f_{A,\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\frac{5}{24}}(q) \;=\; 2f_{A,\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\frac{5}{24}}(q) &= f_{A,\left(\begin{smallmatrix}\beta\\-\beta\end{smallmatrix}\right),\frac{b^2}{4}-\frac{1}{24}}(q) && \text{if } \beta \in \mathbb{Z} \text{ is odd,} \\ f_{A,\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),\frac{5}{24}}(q) &= f_{A,\left(\begin{smallmatrix}\beta\\-\beta\end{smallmatrix}\right),\frac{b^2}{4}-\frac{1}{24}}(q) && \text{if } \beta \in \mathbb{Z} \text{ is even,} \\ f_{A,\frac{1}{2}\left(\begin{smallmatrix}-1\\-1\end{smallmatrix}\right),\frac{1}{48}}(q) &= 2f_{A,\left(\begin{smallmatrix}\beta\\-\beta\end{smallmatrix}\right),\frac{b^2}{4}-\frac{1}{24}}(q) && \text{if } \beta \in \frac{1}{2} + \mathbb{Z}. \end{aligned} \tag{5.23}$$

Since we focus on Nahm sums with $b \in \mathbb{Z}^r$, we can restrict ourselves to the functions

$$\tilde{F}_A(\tau) \;=\; F_A(q) \;=\; \begin{pmatrix} f_{A,\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),-\frac{1}{24}}(q) \\ f_{A,\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right),\frac{5}{24}}(q) \end{pmatrix}, \qquad q = e^{2\pi i \tau}. \tag{5.24}$$

Then $\tilde{F}_A(\tau) = F_A(e^{2\pi i \tau})$ is a vector-valued modular function with

$$\tilde{F}_A(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 \\ 0 & \zeta_{24}^{5} \end{pmatrix} \tilde{F}_A(\tau), \qquad \tilde{F}_A\left(\frac{-1}{\tau}\right) \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tilde{F}(\tau), \tag{5.25}$$

where $\zeta_{24} = e^{2\pi i/12}$. We will describe the 1:1 correspondence between the components of F_A and the solutions of the Nahm equation

$$1 - Q_1 \;=\; Q_1^2 Q_2^{-1}, \qquad 1 - Q_2 \;=\; Q_2^2 Q_1^{-1}. \tag{5.26}$$

The only complex solution of equation (5.26) is given by $Q = (\frac{1}{2}, \frac{1}{2})$ and we have $c(Q) = -\frac{V(Q)}{4\pi^2} = -\frac{1}{24}$. Moreover, for the principal branches of the logarithm we have

$$b(Q) \;=\; \frac{1}{2\pi i}\left(A\begin{pmatrix}\log\frac{1}{2}\\ \log\frac{1}{2}\end{pmatrix} - \begin{pmatrix}\log\frac{1}{2}\\ \log\frac{1}{2}\end{pmatrix}\right) \;=\; \begin{pmatrix}0\\0\end{pmatrix}. \tag{5.27}$$

Hence, the solution $Q = (\frac{1}{2}, \frac{1}{2})$ corresponds to the first entry $f_{A,\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),-\frac{1}{24}}(q)$ of $F_A(q)$.

Therefore, we cannot construct all components of $F_A(q)$ from the set of solutions $\mathcal{Q} \subset \mathbb{C}$. However, in addition to this solution, equation (5.26) also has the formal solutions

$$Q \text{ “}=\text{” } \lim_{x\to\infty} \pm(x, 1-x) \;=\; \pm(\infty, -\infty)$$

with volume $V(Q) = \frac{-5}{24}4\pi^2$ and $c(Q) = -\frac{V(Q)}{4\pi^2} = \frac{5}{24}$. The formal computation, cf. (5.9),

$$\lim_{x\to\infty}\left(A\begin{pmatrix}\log(x)\\ \log(1-x)\end{pmatrix} - \begin{pmatrix}\log(1-x)\\ \log(x)\end{pmatrix}\right) \;=\; \lim_{x\to\infty} 2\begin{pmatrix}\log(x)-\log(1-x)\\ \log(1-x)-\log(x)\end{pmatrix} \;=\; 2\pi i\begin{pmatrix}-1\\1\end{pmatrix} \tag{5.28}$$

justifies $b(\pm(\infty,-\infty)) \;=\; \pm(-1,1)$. Therefore, the solutions $Q = \pm(\infty,-\infty)$ correspond to the second component of $F_A(q) = f_{A,\pm\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right),\frac{5}{48}}(q)$.

We can compute the transformation under $\tau \mapsto \frac{-1}{\tau}$ using equation (5.14). For the solution $Q' = (\frac{1}{2}, \frac{1}{2})$, we have $\delta_A(Q') = 2$ which gives the first column in (5.25). For $Q' = \pm(\infty,-\infty)$, a formal computation suggests with $Q' = \lim_{x\to\infty}(x, 1-x)$ that $\delta_A(Q') = 2$ which gives the second column in (5.25).

5.1.4. Vector-valued modularity for non-even matrices

We briefly discuss how the vector-valued transformation from subsection 5.1.2 generalises to Nahm sums associated with non-even matrices. For this, we also have to take generalised Nahm sums into consideration. For a symmetric, positive definite matrix $A \in \mathbb{Q}^{r\times r}$, $b \in \mathbb{Q}^r$, $c \in \mathbb{Q}$, and a vector $\delta \in \mathbb{C}^r$ consisting of roots of unity, we define the generalised Nahm sum

$$f^{\delta}_{A,b,c}(q) \;:=\; \sum_{n\in\mathbb{Z}^r_{\geq 0}} \delta_1^{n_1}\cdots\delta_r^{n_r}\,\frac{q^{\frac{1}{2}n^TAn+b^Tn+c}}{(q)_{n_1}\cdots(q)_{n_r}}. \tag{5.29}$$

Moreover, for A and δ as above we define the generalised Nahm equation

$$1 - Q_i \;=\; \delta_i\prod_{j=1}^{r} Q_j^{A_ij}, \qquad i = 1,\ldots,r, \tag{5.30}$$

and write $\mathcal{Q}^{(\delta)}_A$ for the set of solutions.

If the generalised Nahm sum $\tilde{f}^{\delta}_{A,b,c}(\tau) = f^{\delta}_{A,b,c}(e^{2\pi i\tau})$ is a modular function, it can be considered as a component of a vector-valued modular function for $\mathrm{SL}_2(\mathbb{Z})$. In this case, the components are in 1:1 correspondence with the solutions of the corresponding Nahm equations (5.30). Then, we have

$$\tilde{f}^{\delta}_{A,b,c}\Big(-\frac{1}{\tau}\Big) \;=\; \sum_{Q\in\mathcal{Q}^{\delta}_A}\frac{\prod_j Q_j^{b_j}}{\delta_A(Q)}\,\tilde{f}^{e(b)}_{A,b(Q),c(Q)}(\tau) \tag{5.31}$$

where $e(b)$ denotes the vector $(e^{2\pi i b_i})_{i=1,\ldots,r}$.

Example 5.1.3. We consider the Nahm sum for $A = 1$. It is known that $f_{1,b,c}$ is exactly modular if $(b, c) \in \{(0, -\frac{1}{48}), (\pm\frac{1}{2}, \frac{1}{24})\}$. However, the space spanned by these function is not invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. Only if we include the signed Nahm sum

$$f^{-}_{1,0,-\frac{1}{48}}(q) \;=\; \sum_{n\geq 0}(-1)^n \frac{q^{\frac{1}{2}n^2-\frac{1}{48}}}{(q)_n}, \tag{5.32}$$

the vector-valued function

$$\tilde{F}_1(\tau) \;=\; F_1(q) \;=\; \begin{pmatrix} f_{1,0,-\frac{1}{48}}(q) \\ f_{1,\frac{1}{2},\frac{1}{24}}(q) \\ f^{-}_{1,0,-\frac{1}{48}}(q) \end{pmatrix} \;=\; \begin{pmatrix} \frac{\eta(\tau)^2}{\eta(\tau/2)\eta(\tau)} \\ \frac{\eta(2\tau)}{\eta(\tau)} \\ \frac{\eta(\tau/2)}{\eta(\tau)} \end{pmatrix}, \qquad q = e^{2\pi i\tau} \tag{5.33}$$

is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. We remark that the function $\tilde{F}_1(\tau)$ consists of the well-known Weber modular functions ([YZ97]). More precisely, $\tilde{F}_1(\tau)$ transforms under $\mathrm{SL}_2(\mathbb{Z})$ via

$$\tilde{F}_1(\tau+1) \;=\; \begin{pmatrix} 0 & 0 & \zeta_{48}^{-1} \\ 0 & \zeta_{48}^{2} & 0 \\ \zeta_{48}^{-1} & 0 & 0 \end{pmatrix} \tilde{F}_1(\tau), \quad \tilde{F}_1\left(\frac{-1}{\tau}\right) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \tilde{F}_1(\tau), \tag{5.34}$$

where $\zeta_{48} = e^{\pi i/24}$.

In order to obtain the 1:1 correspondence, first note that the ordinary Nahm equation $1 - Q = Q$ has a unique solution $Q = \frac{1}{2}$. With $c(Q) = -\frac{V(Q)}{4\pi^2} = -\frac{1}{48}$, the solution $Q = \frac{1}{2}$ corresponds to the Nahm sums $f^{\pm}_{1,0,-\frac{1}{48}}(q)$. The generalised Nahm equation (5.32) is given by

$$1 - Q \;=\; -Q \tag{5.35}$$

with no solution in $\mathbb{C}$ but two formal solutions $\pm\infty$. If we compute the volume of $\pm\infty$, we obtain

$$c(\pm\infty) \;=\; \frac{-V(\pm\infty)}{4\pi^2} \;=\; \frac{1}{24} \tag{5.36}$$

and $b(\pm\infty) = \pm\frac{1}{2}$. Then the functions $f_{1,\pm\frac{1}{2},c}(q)$ are indeed modular. The modular transformation given in (5.34) can be computed using (5.31).

This and other examples suggest that if the matrix A is integral but not even, the appropriate collection of b's that we must take to find associated modular Nahm sums is given by choosing $b_i \in \mathbb{Z}$ if A_{ii} is even and $b_i \in \frac{1}{2}\mathbb{Z}$ and $\delta_i = \pm 1$ if A_{ii} is odd.

5.2. Quantum modularity

In the context of quantum knot invariants, Garoufalidis-Zagier noticed that certain q-series associated to knots possess modular properties, even though they are not modular in a classical sense [GZ23, GZ24]. The new concept of modularity they discovered is called *(holomorphic) quantum modularity.*

Since there is no uniform definition of this concept, and quantum modular forms occur in different shapes, we will only explain the idea. Recall that for a modular function

$f : \mathbb{H} \to \mathbb{C}$ for $\mathrm{SL}_2(\mathbb{Z})$, the quotient $f(\tau)/f(-\frac{1}{\tau})$ is constant. Quantum modularity of a function f means that the failure of modularity $f(\tau)/f(-\frac{1}{\tau})$ is not required to be constant but *more analytic* than f itself. For example, if f is defined on $\mathbb{Q}$, the failure of modularity can have an analytic continuation to $\mathbb{R}$. For a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ *holomorphic* quantum modularity means that the failure of modularity is more analytic than f itself, meaning that it has a holomorphic continuation to some bigger domain than $\mathbb{H}$.

In his thesis, Wheeler [Whe23] proves that all one-dimensional Nahm sums are vector-valued holomorphic quantum modular forms. This means that a holomorphic extension property holds even if the sums $f_{A,b}(q)$ are not modular. We expect that the same is true for higher-dimensional Nahm sums.

Theorem 5.2.1 ([Whe23], Theorem 5)**.** *For $A \in \mathbb{Z}_{>0}$, the function*

$$\tilde{F}_A(\tau) \;=\; F_A(q) := \begin{pmatrix} f_{A,0}(q) \\ f_{A,1}(q) \\ \vdots \\ f_{A,A-1}(q) \end{pmatrix} \tag{5.37}$$

is a vector-valued holomorphic quantum modular form*, in the sense that*

$$\tilde{F}_A\Big(-\frac{1}{\tau}\Big) \;=\; \Omega_A(\tau)\tilde{F}_A(\tau), \tag{5.38}$$

for all $\tau \in \mathbb{H}$ and a function $\Omega_A(\tau) \in \mathrm{Hol}(\mathbb{C} \setminus (-\infty, 0))^{A\times A}$.

If $\tilde{F}_A(\tau)$ is a vector-valued modular function on $\mathrm{SL}_2(\mathbb{Z})$, then of course the matrix $\Omega_A(\tau)$ is constant up Laurent polynomials in rational powers of q and $\tilde{q}$. However, in cases where we expect modularity from Nahm's conjecture, Wheeler's proof does not show that $\Omega_A(\tau)$ is essentially constant.

5.2.1. Example: $A = 4$

We will illustrate how the vector-valued quantum modularity of the non-modular Nahm sum $f_{4,b}(q)$ can be seen numerically by making subleading terms in the asymptotics of $f_{4,b}(q)$, cf. Example 4.3.2, visible. We present computations for the Nahm sum for $f_{4,0}(q)$ but similar computations can certainly be done with all Nahm sums.

For a divergent power series, the Padé approximation gives a rational function that approximates the given series to any desired degree of accuracy. With 120 coefficients of the power series $\Phi^{(Q)}_{4,b}(h)$ as algebraic numbers let $s(\Phi_{4,b}(h))$ denote the [60/60] Padé approximation of $\Phi_{4,b}(h) \in \mathbb{C}[[h]]$.

Numerical Observation 5.2.2. Set

$$\begin{aligned} h \;&=\; \frac{e^{.48\pi i}}{4} \;=\; 0.01569763\cdots + 0.2495067\cdots i, \\ q \;&=\; e^{-h} \;=\; 0.95394\cdots - 0.24308\cdots i. \end{aligned} \tag{5.39}$$

We enumerate the solutions of $1-Q=Q^4$ as in equation (4.18). The values of the Padé approximations of $\Phi_{4,0}^{(Q)}(h)$ are given by

$$\begin{aligned} s(\Phi_{4,0}^{(Q_1)})(h) &\approx 0.3612519-0.0005132i, & s(\Phi_{4,0}^{(Q_2)})(h) &\approx 0.7398696-0.0033125i, \\ s(\Phi_{4,0}^{(Q_3)})(h) &\approx 0.4293409-0.1774570i, & s(\Phi_{4,0}^{(Q_4)})(h) &\approx 0.4454061+0.1710354i. \end{aligned} \tag{5.40}$$

The first approximation of $f_{4,0}(q)\approx 10.15883-1.542461i$ is given by $e^{V_3/h}s(\Phi_{4,0}^{(Q_3)})(h)$ and we have

$$|f_{4,0}(q)-e^{V_3/h}s(\Phi_{4,0}^{(Q_3)})(h)| \approx 0.8398995123, \tag{5.41}$$

see also Example 4.3.2. Figure 4.1 suggests that the next contribution comes from the solution Q_2. Indeed, we can make a better approximation by substracting the contribution for Q_2

$$|f_{4,0}(q)-e^{V_3/h}s(\Phi_{4,0}^{(Q_3)})(h)-e^{V_2/h}s(\Phi_{4,0}^{(Q_2)})(h)| \approx 0.004801327866. \tag{5.42}$$

Similarly, we see subleading terms corresponding to the solutions Q_1 and Q_2 leading to an even better approximation

$$\begin{aligned} &|f_{4,0}(q)-e^{V_3/h}s(\Phi_{4,0}^{(Q_3)})(h)-e^{V_2/h}s(\Phi_{4,0}^{(Q_2)})(h) \\ &\qquad -e^{V_1/h}s(\Phi_{4,0}^{(Q_1)})(h)-e^{(V_4-4\pi^2)/h}s(\Phi_{4,0}^{(Q_1)})(h)| \\ &\approx 2.049142874\times 10^{-9}. \end{aligned} \tag{5.43}$$

The next term again corresponds to Q_2, but now with a different lift of the volume V_2, i.e., a different branch of the dilogarithm shifted by $4\pi^2$

$$\begin{aligned} &|f_{4,0}(q)-e^{V_3/h}s(\Phi_{4,0}^{(Q_3)})(h)-e^{(V_2-8\pi^2)/h}s(\Phi_{4,0}^{(Q_2)})(h) \\ &\qquad -e^{V_1/h}s(\Phi_{4,0}^{(Q_1)})(h)-e^{(V_4-4\pi^2)/h}s(\Phi_{4,0}^{(Q_1)})(h)| \\ &\approx 1.238513219\times 10^{-12}. \end{aligned} \tag{5.44}$$

With $e^{-4\pi^2/h}=\tilde{q}=(4.2814+2.4652i)\times 10^{-5}$, we can continue like this until we end up with the approximation

$$\begin{aligned} f_{4,0}(q) &\approx e^{V_1/h}s(\Phi_{4,0}^{(Q_1)})(h)(1+\tilde{q}^4+\tilde{q}^5) \\ &\quad + e^{V_2/h}s(\Phi_{4,0}^{(Q_2)})(h)(1+\tilde{q}^2+\tilde{q}^3+\tilde{q}^4+\tilde{q}^5+\tilde{q}^6) \\ &\quad + e^{V_3/h}s(\Phi_{4,0}^{(Q_3)})(h)(1+\tilde{q}^3+\tilde{q}^4+\tilde{q}^5+\tilde{q}^6) \\ &\quad + e^{V_4/h}s(\Phi_{4,0}^{(Q_3)})(h)\tilde{q}(1+\tilde{q}^5) \end{aligned} \tag{5.45}$$

with an error of size 7.507881×10^{-29}. We recognise the first coefficients of the (formal) expansion of

$$\begin{aligned} f_{4,0}(q) &\stackrel{?}{=} e^{V_1/h}s(\Phi_{4,0}^{(Q_1)})(h)f_{4,2}(\tilde{q})+e^{V_2/h}s(\Phi_{4,0}^{(Q_2)})(h)f_{4,0}(\tilde{q}) \\ &\quad + e^{V_3/h}s(\Phi_{4,0}^{(Q_3)})(h)f_{4,1}(\tilde{q})+e^{V_4/h}s(\Phi_{4,0}^{(Q_3)})(h)\tilde{q}f_{4,3}(\tilde{q}). \end{aligned} \tag{5.46}$$

It is believed that the components of matrix-valued holomorphic function $\Omega_{A,b}(\frac{ih}{2\pi})$ from Theorem 5.2.1 agree with the resummations $s(\Phi^{(Q)}_{A,b}(h))$ for $Q \in \mathcal{Q}_A$ and $b = 0, \cdots, A-1$. Hence, with the notation

$$F_4(q) \;=\; \begin{pmatrix} f_{4,0}(q) \\ f_{4,1}(q) \\ f_{4,2}(q) \\ f_{4,3}(q) \end{pmatrix}, \tag{5.47}$$

equation (5.46) can be interpreted as an asymptotic version of the holomorphic quantum modularity $\tilde{F}_4(-\frac{1}{\tau}) = \Omega_4(\tau)\tilde{F}_4(\tau)$ of $\tilde{F}_4(\tau)$ from Theorem 5.2.1.

6. Nahm's observation revisited

6.1. Introduction

As discussed in Section 3.3, "Nahm's conjecture" as formulated in [Zag07] states that for a given symmetric, positive definite matrix A, the Nahm sum $f_{A,b,c}(q)$ is modular for some $b \in \mathbb{Q}^r, c \in \mathbb{Q}$ if and only the images of all solutions $\mathcal{Q}_A$ of the Nahm equation vanish in $\mathcal{B}(\mathbb{C})$. In [VZ11], Vlasenko-Zwegers gave counterexamples to this conjecture. They show that the Nahm sum for

$$A = \frac{1}{2}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \qquad b = \frac{1}{2}\begin{pmatrix} -2+\beta \\ 2+\beta \end{pmatrix} \tag{6.1}$$

can be written as a one-dimensional Nahm sum

$$f_{\frac{1}{2}\left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right), \frac{1}{2}\left(\begin{smallmatrix} -2+\beta \\ 2+\beta \end{smallmatrix}\right), \frac{c}{2}}(q^2) = f_{2,\beta,c}(q) \tag{6.2}$$

and is thus modular for $(\beta, c) \in \{(0, -\frac{1}{60}), (1, \frac{11}{60})\}$. The same, of course, is true for $b = \frac{1}{2}\left(\begin{smallmatrix} 2+\beta \\ -2+\beta \end{smallmatrix}\right)$. However, solutions of the corresponding Nahm equation are given by $2[x]$, where x satisfies

$$0 = (x^2 + x - 1)(x^2 - x + 1). \tag{6.3}$$

While the elements $2[x]$ corresponding to $x = \frac{-1\pm\sqrt{5}}{2}$ vanish in $\mathcal{B}(\mathbb{C})$, the elements corresponding to $x = \frac{1\pm\sqrt{-3}}{2}$ do not vanish. This can be proved using the values of $2[x]$ under the Bloch-Wigner dilogarithm and Theorem 2.2.3. In other words, this is a counterexample to the conjecture mentioned above.

Since the matrix $A = \frac{1}{2}\left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$ is not integral, the Nahm equation involves choices of roots. We discuss a related example with an integral matrix A that was also given by Vlasenko-Zwegers.

We will see that for all known counterexamples the corresponding Nahm sums have a representation as a lower-dimensional Nahm sum. This will play a role in the following.

6.2. Integral "counterexamples"

We consider the integral example from [VZ11, p.18], which is modular but whose Nahm equation has some non-torsion solutions in the Bloch group.

We consider the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \tag{6.4}$$

In [VZ11] it has been shown that for $b = \frac{1}{2}(-1+2\beta,\, 1+2\beta,\, 1,\, 1)^T$, $\beta \in \mathbb{Q}$

$$f_{A,b}(q) \;=\; \frac{(q^2;q^2)^2_\infty}{(q;q)^2_\infty}\, f_{\frac{1}{2}\left(\begin{smallmatrix}3&1\\1&3\end{smallmatrix}\right),\frac{1}{2}\left(\begin{smallmatrix}-2+\beta\\2+\beta\end{smallmatrix}\right)}(q^2) \;=\; \frac{(q^2;q^2)^2_\infty}{(q;q)^2_\infty}\, f_{2,\beta}(q), \tag{6.5}$$

where the second equality is equation (6.2). Because A is invariant under permuting the first and the second, as well as the third and fourth, rows and columns, the same identity is true for $b = \frac{1}{2}(1+2\beta,\, -1+2\beta,\, 1,\, 1)^T$, $\beta \in \mathbb{Q}$. In particular, $f_{A,b,c}(q)$ is modular for the following values

b	c	$f_{A,b,c}(q)$
$\frac{1}{2}(-1,\, 1,\, 1,\, 1)^T$	$\frac{1}{15}$	$\frac{\eta(2\tau)^2}{\eta(\tau)^3}\theta_{5,1}(\tau)$
$\frac{1}{2}(1,\, -1,\, 1,\, 1)^T$	$\frac{1}{15}$	$\frac{\eta(2\tau)^2}{\eta(\tau)^3}\theta_{5,1}(\tau)$
$\frac{1}{2}(1,\, 3,\, 1,\, 1)^T$	$\frac{4}{15}$	$\frac{\eta(2\tau)^2}{\eta(\tau)^3}\theta_{5,2}(\tau)$
$\frac{1}{2}(3,\, 1,\, 1,\, 1)^T$	$\frac{4}{15}$	$\frac{\eta(2\tau)^2}{\eta(\tau)^3}\theta_{5,2}(\tau)$

where $\theta_{5,j}(\tau)$ is defined in (2.60).

The Nahm equation (3.8) for A as in (6.4) has eight solutions in two Galois orbits. The first Galois orbit is of cardinality four and given by

$$\Big(u, u, \frac{1}{1+u}, \frac{1}{1+u}\Big), \qquad 1-u^2 = u^4. \tag{6.6}$$

The second Galois orbit, also of cardinality four, is given by

$$\Big(u, -u, \frac{1}{1+u}, \frac{1}{1-u}\Big), \qquad 1-u^2 = -u^4, \tag{6.7}$$

i.e., u is a primitive 12th root of unity. The images of the solutions of the form (6.7) do not vanish in $\mathcal{B}(\mathbb{C})$ because their Bloch-Wigner dilogarithm $D(u)+D(-u)+D(\frac{1}{1+u})+D(\frac{1}{1-u})$ $= \pm 1.014942\cdots$ do not vanish, cf. Theorem 2.2.3.

In view of the asymptotics of Nahm sums on rays in the upper half-plane (Theorem 4.3.1), we consider $V(Q)$ as well as $\Phi^{(Q)}_{A,b}(h)$ for the solutions $Q \in \mathcal{Q}_A$ of type (6.7) with $[Q] \neq 0 \in \mathcal{B}(\mathbb{C})$. For $u = e^{\pi i/6}$, we define the solutions

$$Q \;=\; \Big(u, -u, \frac{1}{1+u}, \frac{1}{1-u}\Big), \qquad Q' \;=\; \Big(-u, u, \frac{1}{1-u}, \frac{1}{1+u}\Big) \tag{6.8}$$

from (6.7). Since Q can be transformed into Q' by permuting the first and second as well as the third and fourth entry and A is invariant under the corresponding row and column permutations, we have

$$V(Q) \;=\; V(Q') \;=\; -\frac{5\pi^2}{12} - D(\zeta_{12}^2)i \;=\; -4.1123\cdots \;-\; 1.0149\cdots i. \tag{6.9}$$

Moreover, we can compute for $b = \frac{1}{2}(1,-1,1,1)^T$

$$\begin{aligned}\Phi^{(Q)}_{A,b}(h) \;&=\; \frac{1}{\sqrt{3-\sqrt{-3}}}\Big(1 + \frac{-3+\sqrt{-3}}{432}h + O(h^2)\Big),\\ \Phi^{(Q')}_{A,b}(h) \;&=\; \frac{-1}{\sqrt{3-\sqrt{-3}}}\Big(1 + \frac{-3+\sqrt{-3}}{432}h + O(h^2)\Big)\end{aligned} \tag{6.10}$$

up to order 1 in h. This suggests that

$$\Phi^{(Q)}_{A,b}(h) \stackrel{?}{=} -\Phi^{(Q')}_{A,b}(h), \tag{6.11}$$

and if this is true, then it means that these asymptotic contributions cancel each other out and do not contribute in the asymptotic expansion of $f_{A,b,c}(q)$ near $q = 1$. In fact, we know that (6.11) must hold since the corresponding $f_{A,b,c}(q)$'s are known to be modular. However, it would be desirable to find a direct proof.

We can make a start as follows. Since Q' can be obtained from Q' by permuting the entries, we can reduce the equality in (6.11). Let P be the permutation matrix $(1,2)(3,4)$ then $Q' = PQ$. Since $PAP = A$, we also have $P\tilde{A}(Q')P = \tilde{A}(Q)$. With the change of variables $t \mapsto Pt$ we obtain

$$\Psi^{(Q')}_{A,b}(Pt,h) = \Psi^{(Q)}_{A,Pb}(t,h), \tag{6.12}$$

where $\Psi^{(Q)}_{A,b}(t,h)$ is defined in (4.10). Then, equation (6.11) is equivalent to

$$\Phi^{(Q)}_{A,b}(h) \stackrel{?}{=} -\Phi^{(Q)}_{A,Pb}(h) = -\Phi^{(Q)}_{A,b+e}(h) \tag{6.13}$$

with $b^tP = b + (-1,\, 1,\, 0,\, 0)^T =: b + e$.

Functional equations We also discuss a different approach for explaining why some solutions are not relevant for the modularity of $f_{A,b,c}(q)$. From (6.5), we see that, in addition to the functional equations in Proposition 3.2.2, elements in the $\mathbb{Z}[q^{\pm}]$-submodule

$$\Big\langle f_{A,b}(q) :\ b = \Big(-\frac{1}{2}+\beta,\, \frac{1}{2}+\beta,\, \frac{1}{2},\, \frac{1}{2}\Big)^T,\, \beta \in \mathbb{Z}\Big\rangle \tag{6.14}$$

fulfil the additional equations

$$\begin{aligned} f_{A,b}(q) - f_{A,b+e_1+e_2}(q) &= \frac{(q^2;q^2)^2_\infty}{(q;q)_\infty}\big(f_{2,\beta}(q) - f_{2,\beta+1}(q)\big) \\ &= \frac{(q^2;q^2)^2_\infty}{(q;q)_\infty}\, q^{1+\beta} f_{2,\beta+2}(q) \\ &= q^{1+\beta} f_{A,b+2e_1+2e_2}(q). \end{aligned} \tag{6.15}$$

A similar functional equation holds for elements in the $\mathbb{Z}[[q^{\pm}]]$-submodule

$$\Big\langle f_{A,b}(q) :\ b = \Big(\frac{1}{2}+\beta,\, -\frac{1}{2}+\beta,\, \frac{1}{2},\, \frac{1}{2}\Big)^T,\, \beta \in \mathbb{Z}\Big\rangle. \tag{6.16}$$

As discussed in Remark 3.2.6, the functional equation suggests that we should also consider the additional Nahm equation

$$1 - Q_1Q_2 = Q_1^2Q_2^2. \tag{6.17}$$

The solutions of the form (6.6) fulfil this equation while the solutions of the form (6.7) do not. This suggests that the Nahm equations associated with the submodule (6.14) have to be completed by equation (6.17).

The main ingredient for the functional equation (6.15) is that the four-dimensional Nahm sum for A as in (6.4) can be written in terms of a one-dimensional Nahm sum in (6.5).

6.3. Modular combinations of non-modular Nahm sums

Even if a Nahm sum $f_{A,b,c}(q)$ for a given matrix A is not modular for any b and c, it is still possible that a linear combination of Nahm sums is modular. Given the functional equation (3.9) of Nahm sums, it is obvious how to construct trivial combinations of modular Nahm sums, namely the constant function 0. Therefore, we are only interested in non-trivial linear combinations of Nahm sums. Several examples of modular linear combinations of non-modular (generalised) Nahm sums are known [AvEH23, KR19]. We prove a new family of examples of such Nahm sums and discuss them in the context of Nahm's problem. Similar to the examples from Section 6.2, we will see that the h-series corresponding to the solutions that do not vanish in $\mathcal{B}(\mathbb{C})$ will not appear in the asymptotic expansion.

6.3.1. An example

We consider the Nahm sum corresponding to $A = \left(\begin{smallmatrix} 8 & 5 \\ 5 & 4 \end{smallmatrix}\right)$. The Nahm equation (3.8) for A is given by

$$1 - Q_1 \;=\; Q_1^8 Q_2^5, \qquad 1 - Q_2 \;=\; Q_1^5 Q_2^4 \tag{6.18}$$

and these equations have eight solutions in two Galois orbits. The first Galois orbit consists of four solutions and is given by

$$\begin{aligned} 0 \;&=\; Q_1^4 + Q_1^3 + 3Q_1^2 - 3Q_1 - 1, \\ Q_2 \;&=\; \frac{1}{5}(37 - 25Q_1 - 6Q_1^2 - 9Q_1^3). \end{aligned} \tag{6.19}$$

These solutions, as well as their volumes, are explicitly given as follows.

Q_1	$-\dfrac{V(Q)}{4\pi^2}$
$\frac{1}{4}\left(-1+\sqrt{5}+\sqrt{14\sqrt{5}-26}\right) = 0.88483\cdots$	$-\dfrac{1}{60}$
$\frac{1}{4}\left(-1+\sqrt{5}-\sqrt{14\sqrt{5}-26}\right) = -0.26680\cdots$	$\dfrac{59}{60}$
$\frac{1}{4}\left(-1-\sqrt{5}-i\sqrt{26+14\sqrt{5}}\right) = -0.80902\cdots - 1.8925\cdots i$	$\dfrac{11}{60}$
$\frac{1}{4}\left(-1-\sqrt{5}+i\sqrt{26+14\sqrt{5}}\right) = -0.80902\cdots + 1.8925\cdots i$	$\dfrac{11}{60}$

(6.20)

Recall that the imaginary parts of the volume $V(Q)$ is given by the Bloch-Wigner dilogarithm. Hence, with Theorem 2.2.3, we conclude that the images of these solutions vanish in $\mathcal{B}(\mathbb{C})$.

The second Galois orbit also consists of four solutions and is given by

$$\begin{aligned} 0 \;&=\; Q_1^4 - Q_1^3 + 3Q_1^2 - 3Q_1 + 1 \\ Q_2 \;&=\; 3Q_1 + Q_1^3. \end{aligned} \tag{6.21}$$

The solutions of this form and their volumes are given by

Q_1	$-\frac{V(Q)}{4\pi^2}$
$0.5520379\cdots \pm 0.2422745\cdots i$	$0.0560814\cdots \pm 0.0301351\cdots i$
$-0.052037\cdots \pm 1.657938\cdots i$	$0.4855853\cdots \pm 0.0428794\cdots i$

(6.22)

Because $\mathrm{Im}(V(Q)) \neq 0$, we deduce that the images of these solutions do not vanish in $\mathcal{B}(\mathbb{C})$.

To sum up, the Nahm equation has both trivial and non-trivial solutions in the Bloch group. While the Nahm sum $f_{A,b}(q)$ is not modular for any $b \in \mathbb{Q}^2$ (this can be proven using the ideas from subsetcion 4.3.2), the linear combinations

$$\begin{aligned} &q^{-1/60}\Big(f_{\left(\begin{smallmatrix}8&5\\5&4\end{smallmatrix}\right),\left(\begin{smallmatrix}-1\\-1\end{smallmatrix}\right)}(q) \;-f_{\left(\begin{smallmatrix}8&5\\5&4\end{smallmatrix}\right),\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)}(q) \;+f_{\left(\begin{smallmatrix}8&5\\5&4\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)}(q)\Big) \\ &\quad=\; f_{2,0,-\frac{1}{60}}(q) \;=\; \frac{\theta_{5,1}(\tau)}{\eta(\tau)}, \\ &q^{11/60}\Big(f_{\left(\begin{smallmatrix}8&5\\5&4\end{smallmatrix}\right),\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(q) \;-f_{\left(\begin{smallmatrix}8&5\\5&4\end{smallmatrix}\right),\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)}(q) \;+f_{\left(\begin{smallmatrix}8&5\\5&4\end{smallmatrix}\right),\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)}(q)\Big) \\ &\quad=\; f_{2,1,\frac{11}{60}}(q) \;=\; \frac{\theta_{5,2}(\tau)}{\eta(\tau)}, \end{aligned} \tag{6.23}$$

where $q = e^{2\pi i\tau}$, are modular, see Example 2.4.6. The equations can be deduced from the following theorem with $\alpha = 2$, $\delta = 1$, and $\beta = 0, 1$. We will discuss the vector-valued modularity of the linear combinations in subsection 6.3.1.

Theorem 6.3.1. *For $\alpha \in \mathbb{Q}_{>0}$, let $A = \left(\begin{smallmatrix}4\alpha & 2\alpha+1\\ 2\alpha+1 & \alpha+2\end{smallmatrix}\right)$. Then for any $\beta \in \mathbb{Q}$ and any root of unity $\delta \in \mathbb{C}$,*

$$f^{\delta}_{\alpha,\beta}(q) \;=\; f^{(1,\delta)}_{A,\left(\begin{smallmatrix}2\beta-1\\ \beta-1\end{smallmatrix}\right)}(q) \;-f^{(1,\delta)}_{A,\left(\begin{smallmatrix}2\beta-1\\ \beta\end{smallmatrix}\right)}(q) \;+f^{(1,\delta)}_{A,\left(\begin{smallmatrix}2\beta\\ \beta\end{smallmatrix}\right)}(q), \tag{6.24}$$

where the Nahm sums $f^{\delta}_{A,b}(q)$ are defined in (5.29).

Proof. We write

$$\begin{aligned} &f^{(1,\delta)}_{A,\left(\begin{smallmatrix}2\beta-1\\ \beta-1\end{smallmatrix}\right)}(q) \;-f^{(1,\delta)}_{A,\left(\begin{smallmatrix}2\beta-1\\ \beta\end{smallmatrix}\right)}(q) \;+f^{(1,\delta)}_{A,\left(\begin{smallmatrix}2\beta\\ \beta\end{smallmatrix}\right)}(q) \\ &\quad=\; \sum_{n,m\geq 0} \delta^m \frac{q^{2\alpha n^2+(\frac{\alpha}{2}+1)m^2+nm(2\alpha+1)+(\beta-1)(2n+m)}}{(q)_n(q)_m}(q^n - q^{n+m} + q^{2n+m}). \end{aligned} \tag{6.25}$$

Using $2\alpha n^2 + (\frac{\alpha}{2}+1)m^2 + nm(2\alpha+1) = \alpha\frac{k^2}{2} + n(k-2n) + (k-2n)^2$ with $k = 2n+m$ we obtain that the combination is equal to

$$\begin{aligned} &\sum_{n,k\in\mathbb{Z}} \delta^k \frac{q^{\alpha\frac{k^2}{2}+n(k-2n)+(k-2n)^2+(\beta-1)k}}{(q)_n(q)_{k-2n}}(q^n - q^{k-n} + q^k) \\ &\quad=\; \sum_{k\in\mathbb{Z}} \delta^k q^{\alpha\frac{k^2}{2}+(\beta-1)k} \sum_{n\in\mathbb{Z}} \frac{q^{n(k-2n)+(k-2n)^2}}{(q)_n(q)_{k-2n}}(q^n - q^{k-n} + q^k). \end{aligned} \tag{6.26}$$

It remains to prove that

$$S(k) \;:=\; \sum_{n\in\mathbb{Z}} \frac{q^{n(k-2n)+(k-2n)^2}}{(q)_n(q)_{k-2n}}(q^n - q^{k-n} + q^k) \;=\; \frac{q^k}{(q)_k} \tag{6.27}$$

for all $k \in \mathbb{Z}$. Using the `qZeil` package in mathematica [PR97] with the line

```
qZeil[q^((k-n)*(k-2*n))/qPochhammer[q,q,n]/qPochhammer[q,q,k-2*n]
  *(q^n+q^k-q^(k-n)),{n,0,Infinity},k,1]
```

gives the recursion

$$S(k) \;=\; \frac{q}{1-q^k} S(k-1). \tag{6.28}$$

One easily checks $S(0) = 1 = 1/(q)_0$ and thus $S(k) = \frac{q^k}{(q)_k}$. $\square$

Asymptotic explanation

We discuss the asymptotic interpretation of the modularity of these linear combination as we did for the "counterexamples" to Nahm's conjecture in Section 6.2. Let $A = \left(\begin{smallmatrix} 8 & 5 \\ 5 & 4 \end{smallmatrix}\right)$ as above. As $h \to 0$ on a ray in the right half plane with $\arg(h) = .44\pi$, we have according to Theorem 4.3.1

$$f_{A,b}(e^{-h}) \;=\; e^{V(Q)/h}\, \Phi^{(Q)}_{A,b}(h) \tag{6.29}$$

for $Q \approx (0.55204 - 0.24227i, 1.7271 - 0.93410i)$ as in (6.22), where

$$V(Q) \;=\; -2.214005098\cdots \;+\; 1.189682374\cdots\, i, \tag{6.30}$$

and $\Phi^{(Q)}_{A,b}(h) \in \overline{\mathbb{Q}}[[h]]$ is defined in (4.12). The first coefficients can be computed explicitly and are given by

$$\begin{aligned}
\Phi^{(Q)}_{A,\left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}\right)}(h) \;=\; \beta_{A,\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)}\Big(&-Q_1^3 + Q_1^2 - 3Q_1 + 2 \\
&+ \frac{-38850547Q_1^3 + 14170261Q_1^2 - 114564270Q_1 + 20496762}{227174412}h \\
&+ \frac{9021651825252Q_1^3 - 2650242944696Q_1^2 + 25438498486713Q_1 - 1890461619879}{450726755175072}h^2 + \cdots\Big),
\end{aligned}$$

$$\begin{aligned}
-\Phi^{(Q)}_{A,\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)}(h) \;=\; \beta_{A,\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)}\Big(&\;\; Q_1^3 - Q_1^2 + 3Q_1 - 3 \\
&+ \frac{52595190Q_1^3 - 19114411Q_1^2 + 149238792Q_1 - 27615300}{227174412}h \\
&+ \frac{-13567527940846Q_1^3 + 136730502036Q_1^2 - 34949018477942Q_1 - 258421848539}{450726755175072}h^2 + \cdots\Big),
\end{aligned}$$

$$\begin{aligned}
\Phi^{(Q)}_{A,\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)}(h) \;=\; \beta_{A,\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)}\Big(&1 \\
&+ \frac{-13744643Q_1^3 + 4944150Q_1^2 - 34674522Q_1 + 7118538}{227174412}h \\
&+ \frac{4545876115594Q_1^3 + 2513512442660Q_1^2 + 9510519991229Q_1 + 2148883468418}{450726755175072}h^2 + \cdots\Big),
\end{aligned} \tag{6.31}$$

where

$$\beta_{A,\binom{0}{0}} \;=\; \frac{1}{\sqrt{11-12Q_1-3Q_1^3}} \;=\; 0.4018796\cdots - 0.1474356\cdots i. \tag{6.32}$$

The coefficients of $\Phi^{(Q)}_{A,b}(h)$ for other Q as in (6.22) are the algebraic conjugates of the coefficients in (6.31), so need not be given explicitly. We see that, at least up to order 3, the sum of these three series vanishes, i.e.,

$$\Phi^{(Q)}_{A,\binom{-1}{-1}}(h) - \Phi^{(Q)}_{A,\binom{-1}{0}}(h) + \Phi^{(Q)}_{A,\binom{0}{0}}(h) \;\stackrel{?}{=}\; 0. \tag{6.33}$$

This equation, if true, implies that the solutions Q as in (6.21) do not appear in the asymptotics of the linear combinations in (6.23) and therefore do not prevent the modularity.

But in fact, as in Section 6.2, equation (6.33) follows from the modularity of the linear combination in (6.23). In order to understand the modularity, it would be desirable to find a proof of the relation (6.33) without using the modularity of the corresponding linear combination.

Vector-valued modularity

From the representation in (6.23), we see that the vector-valued function

$$\begin{aligned}\tilde{F}_A(\tau) \;=\; F_A(q) \;&=\; \begin{pmatrix} q^{-1/60}\left(f_{A,\binom{-1}{-1}}(q) - f_{A,\binom{-1}{0}}(q) + f_{A,\binom{0}{0}}(q)\right) \\ q^{11/60}\left(f_{A,\binom{1}{0}}(q) - f_{A,\binom{1}{1}}(q) + f_{A,\binom{2}{1}}(q)\right)\end{pmatrix} \\ &=\; \begin{pmatrix} f_{2,0,-1/60}(q) \\ f_{2,1,11/60}(q)\end{pmatrix} \;=\; \frac{1}{\eta(\tau)}\begin{pmatrix}\theta_{5,1}(\tau) \\ \theta_{5,2}(\tau)\end{pmatrix}, \qquad q = e^{2\pi i\tau},\end{aligned} \tag{6.34}$$

is a vector-valued modular function under $\mathrm{SL}_2(\mathbb{Z})$. According to (3.5), the transformation is given by

$$\tilde{F}_A(\tau+1) = \begin{pmatrix}\zeta_{60}^{-1} & 0 \\ 0 & \zeta_{60}^{11}\end{pmatrix}\tilde{F}_A(\tau), \qquad \tilde{F}_A\left(\frac{-1}{\tau}\right) \;=\; \frac{2}{\sqrt{5}}\begin{pmatrix}\sin\frac{2\pi}{5} & \sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & -\sin\frac{2\pi}{5}\end{pmatrix}\tilde{F}_A(\tau), \tag{6.35}$$

where $\zeta_{60} = e^{\pi i/30}$.

Following (5.9), we have $c = -\frac{1}{60} = -\frac{V(Q)}{4\pi^2} \bmod 1$ for the first solutions from (6.19). This suggest that the first two solutions Q given there correspond to the first component of the vector-valued function $F_A(q)$, because the power of q in front of the linear combination is given by $c = -\frac{1}{60}$. Similarly, the last two solutions Q in (6.21) correspond to the second component of $F_A(q)$ with $c = \frac{11}{60} = -\frac{V(Q)}{4\pi^2}$.

Therefore, we have a 2:1 correspondence between the solutions of the Nahm equation of the form (6.19) and the components of the vector-valued modular function $F_A(q)$.

Functional equations

Recall the identity

$$f^{(1,\delta)}_{A,\binom{2\beta-1}{\beta-1}}(q) \;-\; f^{(1,\delta)}_{A,\binom{2\beta-1}{\beta}}(q) \;+\; f^{(1,\delta)}_{A,\binom{2\beta}{\beta}}(q) \;=\; f^{\delta}_{\alpha,\beta}(q) \tag{6.36}$$

from Theorem 6.3.1 with $A = \left(\begin{smallmatrix} 4\alpha & 2\alpha+1 \\ 2\alpha+1 & \alpha+2 \end{smallmatrix}\right)$. From the one-dimensional functional equation

$$f^{\delta}_{\alpha,\beta}(q) - f^{\delta}_{\alpha,\beta+1}(q) \;=\; \delta q^{\frac{1}{2}\alpha+\beta} f^{\delta}_{\alpha,\beta+\alpha}(q) \tag{6.37}$$

it follows that, in addition to the standard functional equation of Nahm sums from Proposition 3.2.2, the linear combination of two-dimensional Nahm sums fulfils the recursion

$$\begin{aligned} & f^{(0,\delta)}_{A,\left(\begin{smallmatrix} 2\beta-1 \\ \beta-1 \end{smallmatrix}\right)}(q) \;-f^{(1,\delta)}_{A,\left(\begin{smallmatrix} 2\beta-1 \\ \beta \end{smallmatrix}\right)}(q) \;+f^{(1,\delta)}_{A,\left(\begin{smallmatrix} 2\beta \\ \beta \end{smallmatrix}\right)}(q) \\ & -\left(f^{(1,\delta)}_{A,\left(\begin{smallmatrix} 2\beta+1 \\ \beta \end{smallmatrix}\right)}(q) \;-f^{(1,\delta)}_{A,\left(\begin{smallmatrix} 2\beta+1 \\ \beta+1 \end{smallmatrix}\right)}(q) \;+f^{(1,\delta)}_{A,\left(\begin{smallmatrix} 2\beta+2 \\ \beta+1 \end{smallmatrix}\right)}\right) \\ = \;& \delta q^{\frac{1}{2}\alpha+\beta}\left(f^{(1,\delta)}_{A,\left(\begin{smallmatrix} 2\beta-1+2\alpha \\ \beta-1+\alpha \end{smallmatrix}\right)}(q) \;-f^{(1,\delta)}_{A,\left(\begin{smallmatrix} 2\beta-1+2\alpha \\ \beta+\alpha \end{smallmatrix}\right)}(q) \;+f^{(1,\delta)}_{A,\left(\begin{smallmatrix} 2\beta+2\alpha \\ \beta+\alpha \end{smallmatrix}\right)}(q)\right). \end{aligned} \tag{6.38}$$

Following Remark 3.2.6, the shifts in the b's in the previous equation suggest that we have to take the additional Nahm equation

$$1 - Q_1^2 Q_2 \;=\; Q_1^{2\alpha} Q_2^{\alpha} \tag{6.39}$$

into consideration.

For the case $\alpha = 2$ and $A = \left(\begin{smallmatrix} 8 & 5 \\ 5 & 4 \end{smallmatrix}\right)$ considered above, the previous equation is given by

$$1 - Q_1^2 Q_2 \;=\; Q_1^4 Q_2^2. \tag{6.40}$$

One checks that the solutions of the form (6.19) fulfil this equation while the solutions of the form (6.21) do not. This suggests that for the submodule generated by linear combinations as in Theorem 6.3.1, namely

$$f_{A,\left(\begin{smallmatrix} 2\beta-1 \\ \beta-1 \end{smallmatrix}\right)}(q) \;-f_{A,\left(\begin{smallmatrix} 2\beta-1 \\ \beta \end{smallmatrix}\right)}(q) \;+f_{A,\left(\begin{smallmatrix} 2\beta \\ \beta \end{smallmatrix}\right)}(q), \qquad \beta \in \mathbb{Q}, \tag{6.41}$$

the additional Nahm equation (6.40) should be taken into consideration as well.

As for the example discussed above, the functional equation in (6.38) is based on the fact that the linear combination of two-dimensional Nahm sums in (6.24) can be written as a one-dimensional Nahm sum.

6.3.2. More examples

In [Zag07, p.46], Zagier gives a list of two-dimensional matrices for which the image of the unique solution of the Nahm equation in $(0,1)^2$ vanishes in $\mathcal{B}(\mathbb{C})$. The matrices with other solutions whose images do not vanish in $\mathcal{B}(\mathbb{C})$ are

$$\begin{pmatrix} 4 & 3 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 8 & 3 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 8 & 5 \\ 5 & 4 \end{pmatrix}, \begin{pmatrix} 11 & 9 \\ 9 & 8 \end{pmatrix}, \begin{pmatrix} 24 & 19 \\ 19 & 16 \end{pmatrix}, \begin{pmatrix} \frac{5}{2} & 2 \\ 2 & 2 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 8 & 1 \\ 1 & 2 \end{pmatrix}, \tag{6.42}$$

three infinite families, as well as their permutations and inverses. For these matrices, there are numerically no $b \in \mathbb{Q}^2, c \in \mathbb{Q}$ such that $f_{A,b,c}$ is modular.

From Theorem 6.3.1, we obtain modular linear combination for $\alpha \in \{1/2, 1, 2\}$, i.e.,

$$A \;=\; \begin{pmatrix} 4\alpha & 2\alpha+1 \\ 2\alpha+1 & \alpha+2 \end{pmatrix} \;\in\; \left\{ \begin{pmatrix} 2 & 2 \\ 2 & \frac{5}{2} \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 8 & 5 \\ 5 & 4 \end{pmatrix} \right\}. \tag{6.43}$$

Similarly, Andrews-van Ekeren-Heluani [AvEH23] show that for the matrix $A = \left(\begin{smallmatrix} 8 & 3 \\ 3 & 2 \end{smallmatrix}\right)$, the linear combination

$$f_{A,\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)}(q) - f_{A,\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}(q) + f_{A,\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)}(q) \;=\; \sum_{m \geq 0} \frac{q^{2m^2}}{(q)_{2m}} \tag{6.44}$$

is modular up to a rational power of q. Hence, there are modular linear combinations for four of the "individual" matrices. The matrices for which there is no known modular combinations yet are

$$\begin{pmatrix} 11 & 9 \\ 9 & 8 \end{pmatrix}, \begin{pmatrix} 24 & 19 \\ 19 & 16 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 8 & 1 \\ 1 & 2 \end{pmatrix}, \tag{6.45}$$

the infinite families, and their inverses and permutations. Based on these examples we make the following guess.

Guess 6.3.2. *Let A be a positive definite, symmetric matrix such that some solutions of the Nahm equation* (3.8) *vanish in $\mathcal{B}(\mathbb{C})$. Then there exist non-trivial modular linear combinations of Nahm sums $f_{A,b,c}(q)$.*

The idea behind this expectation is that if, for a given matrix A as above, the solutions of the Nahm equation have different Galois orbits, the $\mathbb{Z}[q^{\pm}]$-module of Nahm sums is expected to have submodules corresponding to the different Galois orbits. If the classes of the corresponding Galois orbit vanish in $\mathcal{B}(\mathbb{C})$, the associated submodule is expected to contain modular functions, e.g., as linear combinations of Nahm sums. However, the submodules corresponding to Galois orbits should also exist if no class of Galois orbits vanishes in $\mathcal{B}(\mathbb{C})$. These ideas are being developed in a paper in preparation with Don Zagier.

Part II.

Knots and q-series

7. The tail of the coloured Jones polynomial

7.1. Introduction

If we intertwine a string and glue the ends together, we obtain a knot. Knots appear in the arts and have several applications in science, for example in the study of DNA, chemistry, and statistical mechanics. The mathematical study of knots began at the end of the nineteenth century and turned out to be crucial for modern physics.

To study knots, knot invariants are helpful. Classical examples include polynomials, such as the Alexander or Jones polynomial, or q-series. In some cases, these q-series turn out to have modular properties. In this part of the thesis, we study a q-series $\Phi_K(q)$ associated with an alternating link K, called the tail of the coloured Jones polynomial. The main result of this part is presented in Section 9.1 and is a general formula for $\Phi_K(q)$ for a class of links in terms of (partial) theta functions. The results presented here, which are the author's work, will also appear in a paper with Robert Osburn [OS24].

In this Chapter, we recall the background on knots, the coloured Jones polynomial and the tail of the coloured Jones polynomial.

7.2. Knots, links, and the (coloured) Jones polynomial

We will give some background on knot theory. References include [Ada94, Lic97].

Definition 7.2.1. A *link of* $m \in \mathbb{Z}_{\geq 1}$ *components* is an embedding of m copies of S^1 into $\mathbb{R}^3$. A link with one component is a *knot*. We say that two links are equivalent if there exists an orientation-preserving piecewise linear homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ that maps the former in the latter link.

We will represent knots, resp. links, using *knot, resp. link, diagrams*, i.e., projections to $\mathbb{R}^2$. Examples of knot diagrams are given in Figure 7.1.

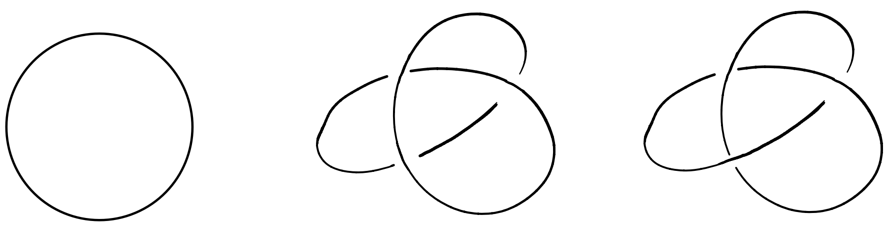


Figure 7.1.: Two diagrams of the unknot and the trefoil knot 3_1

For two link diagrams, the correspong links are equivalent if the diagrams can be transformed into each other by a sequence of *Reidemeister moves* as depicted in Figure 7.2. For example, in Figure 7.1 the second knot diagram can be deformed into the unknot.

A fundamental problem in knot theory is to distinguish knots and links from each other. For example, for the trefoil knot $K = 3_1$, the third knot in Figure 7.1, it is not obvious

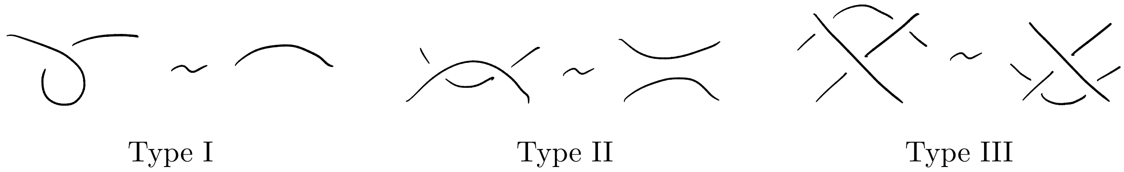


Figure 7.2.: Reidemeister Moves

whether K is equivalent to the unknot. To distinguish knots from each other one can use *knot invariants*, i.e., maps from knot diagrams such that two equivalent diagrams have the same image. For instance, in Example 7.2.3 we will use the Jones polynomial to show that the trefoil knot is not the unknot.

Knots were first tabulated by Tait [Tai00] and Little [Lit00]. It is common to use the notation C_l for the l-th knot with minimal crossing number $C \in \mathbb{Z}_{\geq 2}$ according to a historic (no-canonical) ordering. For example, the trefoil knot, the third example in (7.1) is denoted by 3_1. Another way of tabulating knots is *Conway's notation* [Con70].

We say that a link diagram is alternating if the crossings alternate between over and under crossings along each component of the link. For example, in Figure 7.1 the first and the last diagram are alternating but the second one is not. A link is *alternating* if it has an alternating diagram. As we saw for the unknot in Figure 7.1, an alternating link can also have a non-alternating diagram.

Given two links L_1, L_2, an obvious way to create a new link is by taking the disjoint union of them, denoted by $L_1 \sqcup L_2$. For two knots K_1, K_2 we can form their *composition or connected sum* $K_1 \# K_2$ by removing a small unknotted arc at each knot and connecting each endpoint of a string with one from the other knot. We say that a knot or link is *prime* if it cannot be written as a composition $K_1 \# K_2$ for K_1, K_2 not equivalent to the unknot. Usually, studying invariants reduces to prime knots and links.

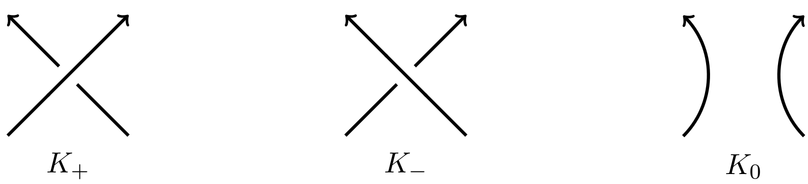
7.2.1. The (coloured) Jones polynomial

We will introduce an important link polynomial called the Jones polynomial that was introduced by Jones in 1985 [Jon85] using von Neumann algebras. The Jones polynomial has been used to prove old conjectures in knot theory and is important for quantum field theory. References include [Lic97, MY18].

We say that a link is *oriented* if every component has an orientation that we indicate by an arrow in the diagram.

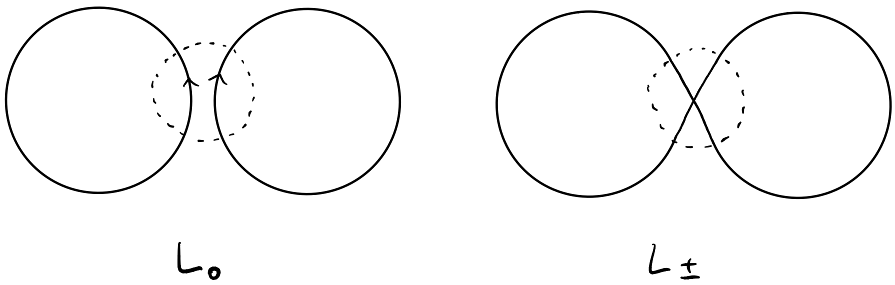
Definition 7.2.2. The *Jones polynomial* $J_K(q) \in \mathbb{Z}[q^{\pm 1/2}]$ is an invariant for oriented links K that satisfies the following.

1. $J_{\bigcirc}(q) = 1$, where $\bigcirc$ denotes the unknot.
2. $J_K(q)$ satisfies
$$(q^{-1/2} - q^{1/2}) J_{K_0}(q) \;=\; q J_{K_-}(q) - q^{-1} J_{K_+}(q), \tag{7.1}$$
where $K_0, K_\pm$ are link diagrams that differ at one crossing by



Equation (7.1) implies that the Jones polynomial is computable in finitely many steps from a given link diagram. One can show that $J_L(q) \in \mathbb{Z}[q^{\pm 1}]$ if L is a link with an odd number of components, e.g., a knot. From (7.1) it follows that $J_L(q)$ does not change if we reverse the orientation of every component of L.

Example 7.2.3. 1. Let L be the link consisting of two disjoint unknots. With $L = L_0$ we consider the "crossing" in the following diagrams

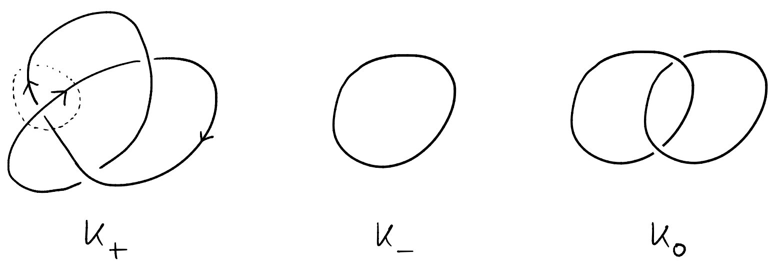


(7.2)

where $L_\pm$ corresponds to the two possible crossings. We note that the associated knots $L_\pm$ are both unknots and thus $J_{L_\pm}(q) = 1$ such that

$$J_L(q) \;=\; \frac{qJ_{L_-}(q) - q^{-1}J_{L_+}(q)}{q^{-1/2} - q^{1/2}} \;=\; -q^{-1/2} - q^{1/2}. \tag{7.3}$$

2. Let $K = 3_1$ be the trefoil knot, see Figure 7.1. Applying the relation to the links $K_+ = K, K_0$, and K_-



(7.4)

we obtain

$$\begin{aligned} J_{K_+}(q) &= q^2 J_{K_-}(q) - q(q^{-1/2} - q^{1/2})J_{K_0}(q) \\ &= q^2 - (q^{1/2} - q^{3/2})J_{K_0}(q). \end{aligned} \tag{7.5}$$

The relation (7.1) for K_0 implies $J_{K_0}(q) = -q^{1/2} - q^{5/2}$. Therefore, we have

$$\begin{aligned} J_K(q) &= q^2 + (q^{1/2} - q^{3/2})(q^{1/2} + q^{5/2}) \\ &= q + q^3 - q^4. \end{aligned} \tag{7.6}$$

In particular, $J_K(q) \neq 1$ and we deduce that $K = 3_1$ is not the unknot.

The Jones polynomial has the following properties.

Proposition 7.2.4. *1. If $L = L' \sqcup \bigcirc$, where $\bigcirc$ denotes the unknot, then $J_L(q) = (-q^{-1/2} - q^{1/2})J_{L'}(q)$. More generally, if $L = L_1 \sqcup L_2$ for links L_1, L_2, then $J_{L_1 \sqcup L_2}(q) = (-q^{-1/2} - q^{1/2})J_{L_1}(q)J_{L_2}(q)$.*

2. For knots K_1, K_2, we have $J_{K_1 \# K_2}(q) = J_{K_1}(q)J_{K_2}(q)$.

3. *Denote by L^* the mirror of L, i.e., the link obtained by inverting all crossings of L. Then $J_L(q) = J_{L^*}(q^{-1})$.*

Proof. Parts 1. and 2. follow by applying the relation (7.1) first to one component and then to the other component. For part 3., note that if we invert each crossing, the roles of L_+ and L_- in (7.1) will interchange. □

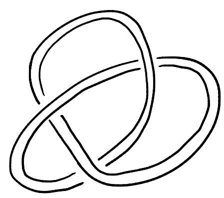
In Example 7.2.3, we computed the Jones polynomial for the trefoil knot 3_1. In particular, we have $J_{3_1}(q) \neq J_{3_1}(q^{-1}) = J_{3_1^*}(q)$, which shows that 3_1 is not equivalent to its mirror 3_1^*.

Even though the Jones polynomial can be used to distinguish knots from each other, it is not a full knot invariant. That means that two links with the same Jones polynomial are not necessarily equivalent. However, it is conjectured that the unknot is the only knot with $J_K(q) = 1$.

We will introduce a generalisation of the Jones polynomial, the *coloured Jones polynomial* [KM91, MY18]. Therefore, define the *Chebyshev polynomials of the second kind* $S_N(x) \in \mathbb{Z}[x]$ for $N \in \mathbb{Z}_{\geq 1}$ recursively by $S_1(x) = 1, S_2(x) = x$ and for $i \geq 2$ via $S_i(x) = xS_{i-1}(x) - S_{i-2}(x)$. For instance,

$$S_3(x) \;=\; x^2 - 1, \qquad S_4(x) \;=\; x^3 - 2x, \qquad S_5(x) \;=\; x^4 - 3x^2 + 1. \tag{7.7}$$

We write $S_N(x) = \sum_{j=0}^{N-1} s_{N,j}x^j$ for $N \in \mathbb{Z}_{\geq 1}$. For a link L denote by $L^{(i)}$ the i-cabling of L, i.e., the link where we replace each component by N copies of the component. For example, a diagram of $3_1^{(2)}$ is given as follows



(7.8)

Definition 7.2.5. For $N \in \mathbb{Z}_{\geq 1}$, the *$N$-th coloured Jones polynomial* $J_K(q;N) \in \mathbb{Z}[q^{\pm 1/2}]$ for an oriented link K is defined by

$$J_K(q;N) \;:=\; \frac{q - q^{-1}}{q^{N/2} - q^{-N/2}} \sum_{j=0}^{N-1} s_{N,j}\, J_{K^{(j)}}(q) \tag{7.9}$$

with the convention $J_{K^{(0)}}(q) = \frac{-1}{q^{-1/2}+q^{1/2}}$.

We have $J_K(q;1) = 1$ as well as $J_K(q;2) = J_K(q)$. Moreover, in our normalisation we have $J_{\bigcirc}(q;N) = 1$ for all $N \in \mathbb{Z}_{\geq 1}$. This can be checked using $J_{\bigcirc^{(j)}} = (-q^{-1/2} - q^{1/2})^{j-1}$ in combination with some well-known identities for the Chebyshev polynomials $S_N(x)$.

Equivalently, the coloured Jones polynomial can be defined for links with m components and a *colouring* of the components [Gar18] or via braid representations of L and the trace of so-called R-matrices [MY18].

Example 7.2.6. For $K = 3_1$, we have $J_K(q) = q + q^3 - q^4$, cf. Example 7.2.3. With (7.8) or using KnotFolio [Mil24], one can compute

$$J_{K^{(2)}}(q) \;=\; -q^{-\frac{23}{2}} + q^{-\frac{21}{2}} + q^{-\frac{17}{2}} - q^{-\frac{9}{2}} - q^{-\frac{5}{2}} - q^{-\frac{1}{2}}. \tag{7.10}$$

Thus, we have with $S_3(x) = x^2 - 1$

$$
\begin{aligned}
J_K(q;3) &= \frac{q+q^{-1}}{q^{3/2}-q^{-3/2}} \left(J_{K^{(2)}}(q) - J_{K^{(0)}}(q)\right) \\
&= q^{-11} - q^{-10} - q^{-9} + q^{-8} - q^{-7} + q^{-5} + q^{-2}.
\end{aligned}
\tag{7.11}
$$

The coloured Jones polynomial fulfils similar properties as the Jones polynomial. The proofs are similar to the proofs in Proposition 7.2.4 or follow directly from them.

Proposition 7.2.7. *1. We have $J_{L_1 \sqcup L_2}(q;N) = (-q^{-1/2} - q^{1/2}) J_{L_1}(q;N) J_{L_2}(q;N)$ for links L_1, L_2 and all $N \geq 1$.*

2. For knots K_1, K_2 we have $J_{K_1 \# K_2}(q;N) = J_{K_1}(q;N) J_{K_2}(q;N)$ for all $N \in \mathbb{Z}_{\geq 1}$.

3. We have $J_L(q;N) = J_{L^*}(q^{-1};N)$ for all $N \in \mathbb{Z}_{\geq 1}$.

The coloured Jones polynomial is a stronger invariant than the Jones polynomial. For example, for $K = 10a_{669}$ the Jones polynomial

$$
\begin{aligned}
J_K(q) &= -q^{-6} + 2q^{-5} - 4q^{-4} + 6q^{-3} - 7q^{-2} + 9q^{-1} - 9 + 9q - 7q^2 + 6q^3 - 4q^4 + 2q^5 - q^6 \\
&= J_K(q^{-1}) = J_{K^*}(q)
\end{aligned}
\tag{7.12}
$$

is palindromic, meaning it cannot distinguish K from its miror K^*. However, the third coloured Jones polynomial [Gar11]

$$
J_K(q;3) = \frac{1}{q^{17}} - \frac{2}{q^{16}} + \cdots - 2q^{18} + q^{19} \neq J_K(q^{-1};N) = J_{K^*}(q;3) \tag{7.13}
$$

is not palindromic and thus $K \neq K^*$.

It is well-known that several other knot invariants are encoded in the coloured Jones polynomial. According to the now proven Melvin-Morton-Rozansky Conjecture [BNG96], the Alexander polynomial $\Delta_K(q) \in \mathbb{Z}[q]$ of a knot K is determined by its coloured Jones polynomials: With the expansion

$$
\frac{J_K(e^h;N)}{J_{\bigcirc}(e^h;N)} = \sum_{i \geq j \geq 0} a_{K,ij} n^j h^i \in \mathbb{Q}[[n,h]] \tag{7.14}
$$

we have

$$
\sum_{i=0}^{\infty} a_{K,ij} h^i = \frac{1}{\Delta_K(e^h)} \in \mathbb{Q}[[h]]. \tag{7.15}
$$

Moreover, the coloured Jones polynomial $J_K(q;N)$ specialised to $q = \zeta_N = e^{2\pi i/N}$ equals another knot invariant, the Kashaev invariant $\langle K \rangle_N \in \mathbb{Z}[[\zeta_N]]$. According to the famous Volume conjecture, it is believed that the Kashaev invariant, and thus the coloured Jones polynomial, contains information about the hyperbolic structure of the knot complement [Kas97, MM01, MMO+02].

Conjecture 7.2.8 (Volume conjecture). *For any knot K we have*

$$
\lim_{N \to \infty} 2\pi \frac{\log J_K(e^{2\pi i/N};N)}{N} = v(K) \tag{7.16}
$$

where $v(K)$ denotes the complexified volume *of $S^3 \setminus K$.*

The complexified volume is the volume as defined in Proposition 2.2.7 for an element in the Bloch group associated to the knot K.

The Volume conjecture is only known to be true for a few knots. It has been refined to higher terms in the asymptotic expansion in (7.16), cf. [DG13, DGLZ09, Gar08, Zag10]. By setting $q = e^{2\pi i/N}$ in equation (7.16), we can rewrite (7.16) as

$$J_K(q;N) \;=\; \tilde{q}^{v(K)i/4\pi^2}(a + O(\tfrac{1}{N})), \qquad N \to \infty \tag{7.17}$$

for some $a \in \mathbb{C}$ and $\tilde{q} = e^{-2\pi i N}$ which resembles the asymptotics of a modular function cf. Proposition 2.4.5. Recently, Garoufalidis-Zagier [GZ24] refined the asymptotics by including subleading exponential terms. This sheds light on the underlying modular behaviour of the Kashaev invariant and thus the coloured Jones polynomial. Moreover, Garoufalidis-Zagier [GZ23] relate the Jones polynomial to q-series that are generalised Nahm sums for half-symplectic matrices.

In the following, we will examine further modular behaviour of limits of $J_K(q,N)$ as $N \to \infty$, with a fixed variable q. This limit is slightly different than the one in the Volume conjecture. We will compare them in subsection 9.4.3.

7.3. Stability properties of the coloured Jones polynomial

We say that a sequence of polynomials $p_n(q) \in \mathbb{Z}[q^{\pm}]$ *stabilises* to a q-series $p(q) \in 1 + \mathbb{Z}[[q]]$ if the first n terms in $p_n(q)$ agree with the first n terms in $p(q)$.

Theorem 7.3.1 ([Arm14, GL15])**.** *Let K be an alternating link. Then the sequence of coloured Jones polynomials $(J_K(q;N))_N$ stabilises to a q-series $\Phi_K(q) \in 1 + \mathbb{Z}[[q]]$, the* tail of the coloured Jones polynomial*.*

For a link L we denote by $(\widehat{J_K}(q;N))_N$ the normalised coloured Jones polynomial, meaning $\widehat{J_K}(q;N) = q^{c_N} J_K(q;N) \in 1 + q\mathbb{Z}[q]$ for some $c_N \in \frac{1}{2}\mathbb{Z}$. Then it follows that $\lim_{N\to\infty} \widehat{J_K}(q;N) = \Phi_K(q)$

Higher order stability, e.g., stability of the sequence $(\widehat{J_K}(q;N) - \Phi_K(q))_N$, was discussed by Garoufalidis-Lê [GL15] and stability for a more general class of knots, so-called adequate knots, was proven in [Arm13]. In the following we will assume that K is alternating.

Example 7.3.2. For $K = 5_2$, the coloured Jones polynomials are given by ([LO19])

$$J_K(q,N) \;=\; q^{N-1} \sum_{0\le n_1 \le n_2 \le N-1} q^{N(N-1-n_2)+n_1(n_1+1)} \frac{(q^{1+N};q)_{n_2}(q^{N-n_2};q)_{n_2}(q;q)_{n_2}}{(q;q)_{n_1}(q;q)_{n_1}(q;q)_{n_2-n_1}} \tag{7.18}$$

and the normalised coloured Jones polynomials are thus given by

$$\widehat{J_K}(q,N) \;=\; q^{1-N} J_K(q,N). \tag{7.19}$$

For instance, we have

$$
\begin{aligned}
\widehat{J_K}(q,1) &= 1,\\
\widehat{J_K}(q,2) &= 1 - q + 2q^2 - q^3 + q^4 - q^5,\\
\widehat{J_K}(q,3) &= 1 - q + 0q^2 + 3q^3 - 2q^4 - q^5 + 4q^6 - 3q^7 - q^8 + \cdots + q^{15},\\
\widehat{J_K}(q,4) &= 1 - q + 0q^2 + q^3 + 2q^4 - 2q^5 - 2q^6 + 2q^7 + 4q^8 + \cdots + q^{30},\\
\widehat{J_K}(q,5) &= 1 - q + 0q^2 + q^3 + 0q^4 + 2q^5 - 3q^6 - q^7 + 2q^8 + \cdots + q^{50},\\
\widehat{J_K}(q,6) &= 1 - q + 0q^2 + q^3 + 0q^4 + 0q^5 + q^6 - 2q^7 - q^8 + \cdots + q^{75},\\
\widehat{J_K}(q,7) &= 1 - q + 0q^2 + q^3 + 0q^4 + 0q^5 - q^6 + 2q^7 - 2q^8 + \cdots + q^{105}.
\end{aligned}
\tag{7.20}
$$

In (7.18), we see that only the summands with $n_2 = N-1$ do not vanish as $N \to \infty$ and thus the (normalised) coloured Jones polynomials stabilise to the q-series

$$
\begin{aligned}
\Phi_K(q) &= \lim_{N\to\infty} \widehat{J_K}(q,N) = (q;q)_\infty \sum_{n_1 \geq 0} \frac{q^{n_1(n_1+1)}}{(q;q)_{n_1}^2} = \sum_{k\geq 0} (-1)^k q^{k(k+1)/2}\\
&= 1 - q + q^3 - q^6 + q^{10} - q^{15} + q^{21} - q^{28} + q^{36} - q^{45} + O(q^{50})
\end{aligned}
\tag{7.21}
$$

where the last equality in the first line follows from Entry 9 in [Ber91].

It follows directly from Proposition 7.2.7 that for links L_1, L_2 we have

$$\Phi_{L_1 \sqcup L_2}(q) = (-q^{-1/2} - q^{1/2})\Phi_{L_1}(q)\Phi_{L_2}(q) \tag{7.22}$$

and for knots K_1, K_2

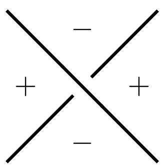
$$\Phi_{K_1 \# K_2}(q) = \Phi_{K_1}(q)\Phi_{K_2}(q). \tag{7.23}$$

7.4. Computation of the tail of the coloured Jones polynomial

The tail of the coloured Jones polynomial for an alternating link diagram has an explicit representation as a q-hypergeometric series. We will recall this representation from [GL15].

7.4.1. Tait graphs

A sign-coloured diagram is a diagram where the faces are coloured with $+$ or $-$, subject to



at each crossing. In other conventions, sign-coloured diagrams are called checkerboard-coloured diagrams and the $+$-coloured, resp. $-$-coloured, faces correspond to black, resp. white, faces. If a link K is alternating, then it has a sign-coloured diagram.

The *Tait graphs* $\mathcal{T}_\pm$ of a sign-coloured diagram D are the graphs with vertices corresponding to the $\pm$-coloured faces of D. Two vertices form an edge if the corresponding faces share a crossing. The $+$–Tait graph for K is the $-$–Tait graph for the mirror K^* of K and vice versa.

7.4.2. The tail of the coloured Jones polynomial

Let $\mathcal{F}$ be the faces of $\mathcal{T}_+$ and $\mathcal{W}$ be the vertices of $\mathcal{T}_+$. For a face $f \in \mathcal{F}$ denote by $v(f)$ the number of adjacent vertices and for vertices $v, w \in \mathcal{W}$ we write $e(v,w)$ for the number of edges from v to w. Moreover, we write $v \in f$ for $v \in \mathcal{W}$ and $f \in \mathcal{F}$ if v is at the boundary of the face f.

We define a quadratic, integral matrix Q, indexed by $\mathcal{F} \cup \mathcal{W}$, in block form

$$Q \;=\; \begin{pmatrix} Q^{\mathcal{F}\mathcal{F}} & Q^{\mathcal{F}\mathcal{W}} \\ Q^{\mathcal{W}\mathcal{F}} & Q^{\mathcal{W}\mathcal{W}} \end{pmatrix}, \tag{7.24}$$

where for $f, g \in \mathcal{F}$ and $v, w \in \mathcal{V}$

$$\begin{aligned} Q^{\mathcal{F}\mathcal{F}}_{f,g} &= \begin{cases} 0 & \text{if } i \neq j, \\ v(f) & \text{if } f = g, \end{cases} & Q^{\mathcal{F}\mathcal{W}}_{fv} &= \begin{cases} 0 & \text{if } v \notin f, \\ 1 & \text{if } v \in f, \end{cases} \\ Q^{\mathcal{W}\mathcal{F}}_{vf} &= Q^{\mathcal{F}\mathcal{W}}_{fv}, & Q^{\mathcal{W}\mathcal{W}}_{vw} &= e(v,w). \end{aligned} \tag{7.25}$$

Moreover, we define a vector $b \in \frac{1}{2}\mathbb{Z}^{|\mathcal{F}|+|\mathcal{W}|}$ indexed by $\mathcal{F} \cup \mathcal{V}$ with entries

$$b_f \;=\; \frac{v(f)}{2} - 1 \text{ for } f \in \mathcal{F}, \qquad b_v \;=\; 1 \text{ for } v \in \mathcal{W}. \tag{7.26}$$

We will assign a variable to each face and each vertex of $\mathcal{T}$. Therefore, we will use the names of the vertices and faces as variable names. In other words, we write $s = (s^{\mathcal{F}}, s^{\mathcal{W}}) \in \mathbb{Z}^{r+s}$ with entries $s^{\mathcal{F}} = (f)_{f \in \mathcal{F}}$, $s^{\mathcal{W}} = (v)_{v \in \mathcal{W}}$.

Moreover, we pick $v_0 \in \mathcal{W}$ with $v_0 \in f_\infty$ and set $v_0 = f_\infty = 0$ where f_∞ denotes the unbounded face. We say that $s \in \mathbb{Z}^{r+s}$ as above is admissible if $f + v \geq 0$ for all $(f,v) \in \mathcal{F} \times \mathcal{V}$ with $v \in f$. In particular, we require $v \geq 0$ whenever $v \in f_\infty$. We denote the set of admissible elements by $\Lambda \subset \mathbb{Z}^{r+s}$.

Theorem 7.4.1. *[GL15, Theorem 1.10] Let K be an alternating link diagram with c crossings. Then the tail of the coloured Jones polynomial of K is given by*

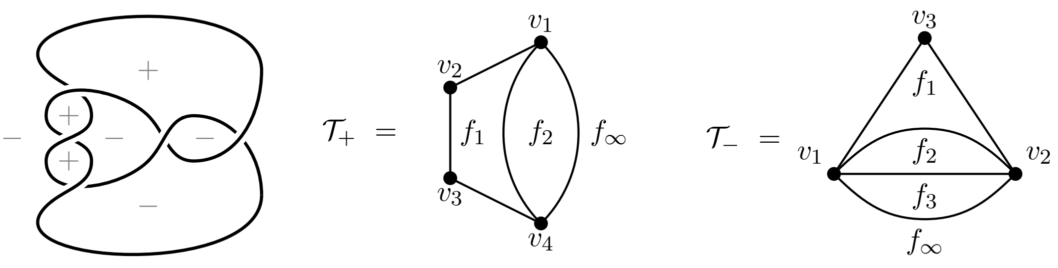
$$\Phi_K(q) \;=\; (q)_\infty^c \sum_{s \in \Lambda} (-1)^{2b^T s} \frac{q^{s^T Q s/2 + b^T s}}{\prod_{v \in f} (q)_{v+f}}, \tag{7.27}$$

where the product is over all pairs $(f,v) \in \mathcal{F} \times \mathcal{W}$ with $v \in f$.

The tail of the Jones polynomial of an alternating link K is already determined by the reduced Tait graph $\mathcal{T}_+$ of K ([GL15, GV15]).

Since Φ_K only depends on the (reduced) Tait graph $\mathcal{T}_+$ of K we also write $\Phi_{\mathcal{T}_+}$ for Φ_K. We illustrate this construction of $\Phi_K(q)$ for $K = 5_2$ and $K = 5_2^*$.

Example 7.4.2. A sign-colouring of $K = 5_2$ and the corresponding Tait graphs are given by



- For $K = 5_2$, the matrix $Q \in \mathbb{Z}^{7\times 7}$ and the vector $b \in \mathbb{Z}^7$ are indexed corresponding to the faces and vertices of $\mathcal{T}_+$ by $(f_1, f_2, f_\infty, v_1, v_2, v_3, v_4)$ and given by

$$Q = \left(\begin{array}{ccc|cccc} 4 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 4 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 0 & 1 & 0 \end{array}\right), \qquad b = \left(\begin{array}{c} 1 \\ 0 \\ 1 \\ \hline 1 \\ 1 \\ 1 \\ 1 \end{array}\right). \tag{7.28}$$

With $f_\infty = v_1 = 0$, an element $s = (f_1, f_2, f_\infty, v_1, v_2, v_3, v_4) \in \mathbb{Z}^7$ is admissible if $f_1, f_2, v_2, v_3, v_4 \geq 0$ and according to Theorem 7.4.1 we have

$$\begin{aligned} \Phi_K(q) &= (q)_\infty^5 \sum_{s\in\Lambda} (-1)^{2b^T s} \frac{q^{s^T Qs/2 + b^T s}}{\prod\limits_{v\in f} (q)_{s_v+s_f}} \\ &= (q)_\infty^5 \sum_{\substack{f_1,f_2\geq 0 \\ v_2,v_3,v_4\geq 0}} \frac{q^{2f_1^2+f_1v_2+f_1v_3+f_1v_4+f_2^2+f_2v_4+v_2v_3+v_3v_4+f_1+v_2+v_3+v_4}}{(q)_{f_1}(q)_{f_1+v_2}(q)_{f_1+v_3}(q)_{f_1+v_4}(q)_{f_2}(q)_{f_2+v_4}(q)_{v_2}(q)_{v_3}(q)_{v_4}} \\ &= 1 - q + q^3 - q^6 + q^{10} - q^{15} + q^{21} + -q^{28} + q^{36} - q^{45} + \mathrm{O}(q^{55}). \end{aligned} \tag{7.29}$$

- For $K^* = 5_2^*$, the matrix $Q \in \mathbb{Z}^{7\times 7}$ and the vector $b \in \mathbb{Z}^7$ are indexed corresponding to the faces and vertices of $\mathcal{T}_-$ by $(f_1, f_2, f_3, f_\infty, v_1, v_2, v_3)$ and given by

$$Q = \left(\begin{array}{cccc|ccc} 3 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 & 3 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right), \qquad b = \left(\begin{array}{c} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \\ \hline 1 \\ 1 \\ 1 \end{array}\right). \tag{7.30}$$

With $f_\infty = v_1 = 0$, an element $s = (f_1, f_2, f_3, f_\infty, v_1, v_2, v_3) \in \mathbb{Z}^7$ is admissible if

$f_1, f_2, f_3, v_2, v_3 \geq 0$ and Theorem 7.4.1 yields

$$\begin{aligned}\Phi_{K^*}(q) &= (q)_\infty^5 \sum_{s\in\Lambda} (-1)^{2b^T s} \frac{q^{s^T Qs/2+b^T s}}{\prod_{v\in f} (q)_{s_v+s_f}} \\ &= (q)_\infty^5 \sum_{\substack{f_1,f_2,f_3\geq 0\\ v_2,v_3\geq 0}} \frac{(-1)^{f_1} q^{\frac{3}{2}f_1^2+f_1v_2+f_1v_3f_2^2+f_2v_2+f_3^2+f_3v_2+v_2v_3+\frac{1}{2}f_1+v_2+v_3}}{(q)_{f_1}(q)_{f_1+v_2}(q)_{f_1+v_3}(q)_{f_2}(q)_{f_2+v_2}(q)_{f_3}(q)_{f_3+v_2}(q)_{v_2}(q)_{v_3}} \\ &= 1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}-q^{35}+q^{40}+O(q^{51}).\end{aligned} \tag{7.31}$$

8. Arborescent knots and links

8.1. The construction of arborescent links

Following [BS10, AFH$^+$21], we will introduce a class of links, called *arborescent links*, that are constructed from a weighted tree. The construction of arborescent links is equivalent to Conway's construction [Con70] of algebraic links. In Section 9.1, we will find a closed formula for the tail of the coloured Jones polynomial for a class of arborescent links.

Definition 8.1.1. A *section-weighted tree* $\Gamma = (\mathcal{V}, \mathcal{E}, w)$ is a planar embedding of a tree $(\mathcal{V}, \mathcal{E})$ together with weights $w(w, v, w')$ for neighbouring vertices w, w' adjacent to $v \in \mathcal{V}$.

For a given section-weighted tree, the weights are depicted as numbers written in the sections around the vertices. We use the convention that we omit all weights 0.

Let $\Gamma = (\mathcal{V}, \mathcal{E}, w)$ be a section-weighted tree. For every vertex, we will associate a twisted ribbon and the structure of the tree will define how the surfaces are plumbed to each other. The boundary of the constructed surface is a link, called the link associated to Γ.

1. Let $v \in \mathcal{V}$ be a vertex with $n \in \mathbb{Z}_{\geq 0}$ adjacent vertices $v_1, \ldots, v_n \in \mathcal{V}$ in counterclockwise order around v. The ribbon associated to v has n marked squares corresponding to v_i, $i = 1, \ldots, n$, and between the squares for v_i and v_{i+1} there are $w(v_i, v, v_{i+1})$ half twists, see Figure 8.1. We use the convention that $v_{n+1} = v_1$ and that is a positive half-twist. The ribbon has two orientations: a horizontal core orientation

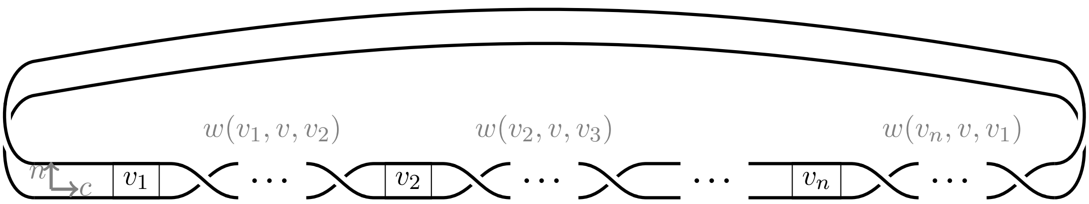


Figure 8.1.: The ribbon associated to v

   (c) and a vertical normal orientation (n).

2. For every edge $(v, v') \in \mathcal{E}$, we plumb the ribbons for v and v' along the squares for v' and v in such a way that the core orientation of v matches the normal orientation of v' and vice versa, cf. Figure 8.2.

3. The plumbed ribbons define a surface and the boundary of this surface is a link L. We say that L is the link associated to Γ.

We consider explicit examples to illustrate the construction.

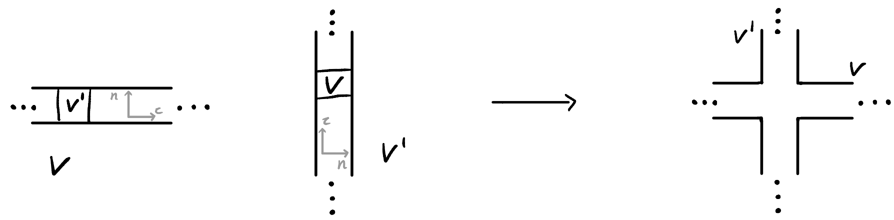
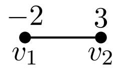


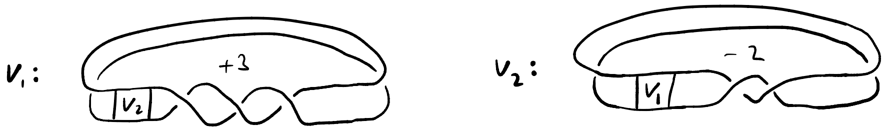
Figure 8.2.: The plumbing of v and v'

Example 8.1.2. 1. For the section-weighted tree

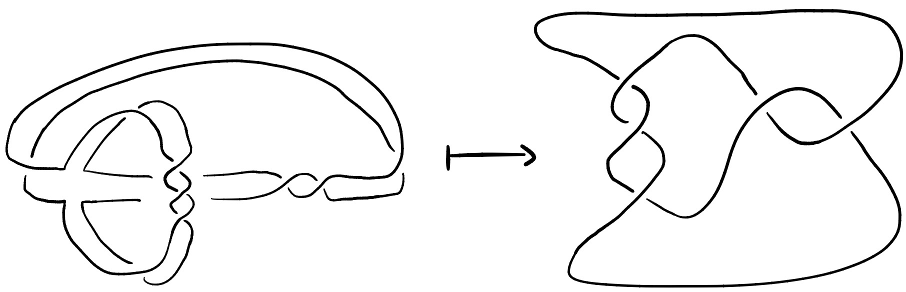


(8.1)

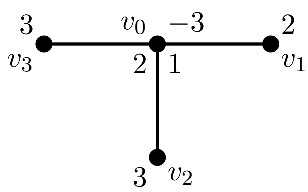
we construct two ribbons associated to the vertices



Plumbing the two ribbons yields the arborescent link $K = 5_2$ that we have already seen in Example 7.4.2.

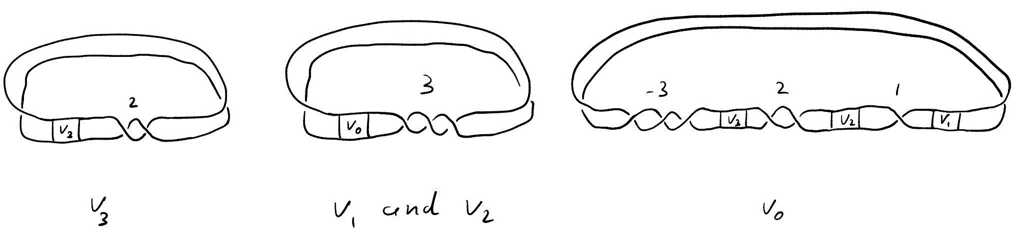


2. We consider the section-weighted tree

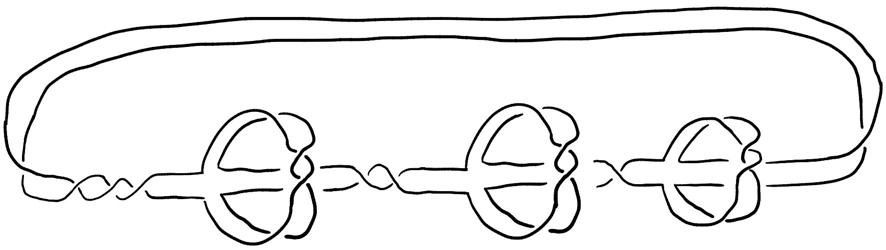


(8.2)

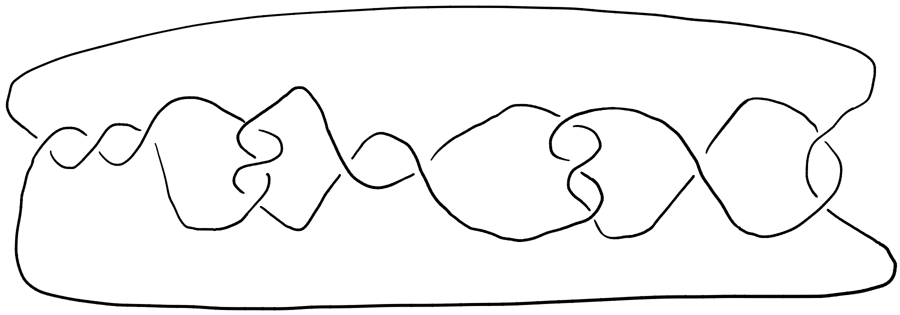
The ribbons corresponding to the vertices $v_0, \ldots, v_3$ are given by



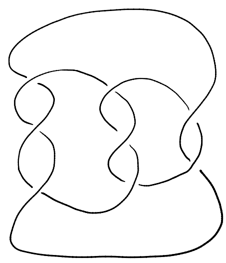
According to the tree, the plumbed ribbons are given by



Their surface can be transformed into the knot



that can be transformed into the knot $K = 8_5$ using a sequence of Reidemeister moves into the knot



(8.3)

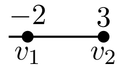
We will see another description of 8_5 as arborescent knot in (8.7). We will see how to transform the corresponding trees into each other in Lemma 8.3.1.

8.2. Arborescent tangles

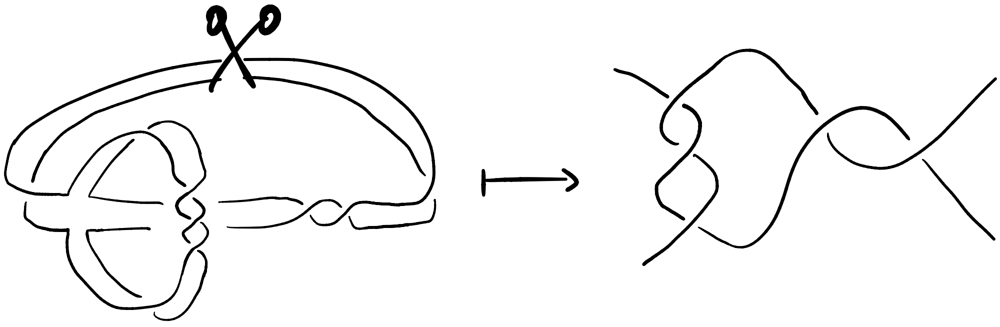
A tangle is defined as a region of a knot or link diagram with exactly 4 emerging strings in the directions NW, NE, SE, and SW. Two tangles are considered to be the same if they are connected via a sequence of Reidemeister moves.

We define the tangle associated to a section-weighted rooted tree. A *section-weighted, rooted tree* is a section-weighted tree with a marked vertex, the *root*, which has an emanating germ in one direction (here depicted by). For a given section-weighted, rooted tree with root v_0 we define the associated tangle as the boundary of the associated surface where the ribbon corresponding to v_0 is cut at the place corresponding to the germ leaving 4 emanating strings.

Example 8.2.1. We consider the weighted tree from Example 8.1.2 with root v_1



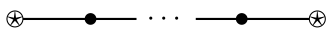
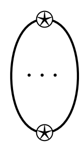
The corresponding tangle is given by

 (8.4)

The $\pm$–Tait graphs for a tangle are defined similarly with the addition of marked vertices corresponding to the North and South faces or the East and West faces. They are depicted by $\circledast$.

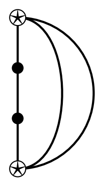
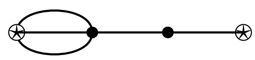
The reduced Tait graphs $\mathcal{T}'_\pm$ of a tangle are obtained from the regular Tait graphs by replacing multiple edges with single edges and removing loops.

Example 8.2.2. 1. Consider the tangle corresponding to the section-weighted, rooted tree $\overset{v}{-\!\bullet}$. With $\pm = \mathrm{sign}(v)$, the Tait graphs are given by

$\mathcal{T}_\pm$ =  $\mathcal{T}_\mp$ =  (8.5)

with $|w|$ vertices, resp. $|w|$ edges.

2. The Tait graphs for the tangle in (8.4) are given by

$\mathcal{T}_+$ =  $\mathcal{T}_-$ =  (8.6)

In the following figures we depict tangles by circles, arborescent trees by squares, and Tait graphs by hexagons.

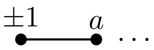
8.3. Moves on weighted planar trees

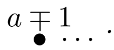
There are moves that change a section-weighted (rooted) tree without changing its associated link or tangle. We mention the following two moves which follow from the construction, see e.g., [BS10, §12.3].

Lemma 8.3.1. *In a section-weighted (rooted) tree we can replace*
$\cdots \underset{b}{\overset{a}{\bullet}}\!\!- \cdots$ *by* $\cdots \underset{b\pm 1}{\overset{a\mp 1}{\bullet}}\!\!- \cdots$ *without changing the corresponding tangle if we reverse the order of all vertices lying on the right of the vertex with odd distance.*

Hence, we can assume that for every vertex only one weight is non-zero. In this case, we write $w(v)$ for the unique non-zero weight of $v \in \mathcal{V}$. In particular, each section-weighted tree can be deformed into a *weighted tree*, meaning that each vertex has at most one non-zero weight.

Lemma 8.3.2. *We can apply the following moves to a section-weighted tree without changing the associated links.*

1. *A subgraph of the form* 

can be replaced by 

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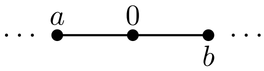
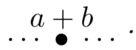
2. *A subgraph of the form* 

can be replaced by 

.

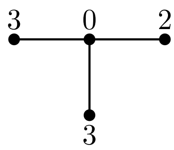
3. *A subgraph of the form* 

can be replaced by 

.

These moves also appear as Kirby-Neumann moves in the context of Dehn surgery and plumbing graphs for 3-manifolds [Neu81].

For example, the tree in (8.2) can be transformed into the tree



(8.7)

giving directly the representation (8.3) of the knot 8_5 .

8.3.1. Alternating weighted trees

Let Γ be a weighted tree such that there exists a bipartition $\mathcal{V}_+ \cup \mathcal{V}_+$ of $\mathcal{V}$ with $\pm w(\mathcal{V}_\pm) \geq 0$. We call a weighted graph with this property *alternating* and it is easy to show that the corresponding link is alternating.

8.3.2. Unsplittable and prime arborescent links

Recall that a link is *unsplittable* if there is no 2–dimensional sphere that splits K. Moreover, a link K is *prime* if it cannot be written as the connected sum of two links. The following Proposition follows from [AFH$^+$21, §17.5.3].

Proposition 8.3.3. *Let $\Gamma = (\mathcal{V}, \mathcal{E}, w)$ be a reduced, weighted tree, meaning it has no vertex with degree ≤ 1 of weight 0. Then the associated link K is prime and unsplittable.*

9. The modularity of the tail of the coloured Jones polynomial

9.1. The main result

In this chapter we present the main result of this part: A formula for the tail of the coloured Jones polynomial in terms of (partial) theta functions for a class of arborescent links.

In [GL15, Appendix D], Garoufalidis and Lê with Zagier conjectured representations of $\Phi_K(q)$ for several knots with up to 8 crossings in terms of products of the functions

$$h_b(q) \;=\; \sum_{n\in\mathbb{Z}} \epsilon_b(n) q^{bn(n+1)/2-n}, \qquad b \in \mathbb{Z}_{\geq 1} \tag{9.1}$$

where

$$\epsilon_b(n) \;=\; \begin{cases} (-1)^n & \text{if } b \text{ is odd,} \\ 1 & \text{if } b \text{ is even and } n \geq 0, \\ -1 & \text{if } b \text{ is even and } n < 0. \end{cases} \tag{9.2}$$

The functions $q^{(2-b)^2/8b} h_b(q)$ are theta functions if b is odd and partial theta functions if b is even. For example, we have

$$h_1(q) \;=\; 0, \qquad h_2(q) \;=\; 1, \qquad h_3(q) \;=\; (q)_\infty, \tag{9.3}$$

$$h_4(q) \;=\; \sum_{k\geq 0} (-1)^k q^{k(k+1)/2}, \qquad h_5(q) \;=\; \vartheta_{5,2}(\tau), \tag{9.4}$$

see (7.21) and (3.2). As in Example 2.4.6, one can show that the theta function $h_b(q)$ for odd b are modular forms. Partial theta functions are known to be related to mock theta functions and possess (quantum) modular behaviour [BM15, Bri21, GO21, Zag09].

The identities for 3_1, 4_1, and 6_3 have been proven by Andrews [And13] and all other identities by Keilthy and Osburn [KO16]. Beirne and Osburn [BO17, Bei20] extended the list to several knots with up to 10 crossings. Some identities are tabulated in Table 9.1. However, identities for a few knots such as $K = 8_5$ are missing, marked with "?" in the table.

As discussed above, $\Phi_K(q)$ is already uniquely determined by the (reduced) Tait graph $\mathcal{T}'$ of K. The most important planar graphs are polygons. The associated q-series were computed in [AD11, Theorem 3.7] using the Andrews-Gordon identities [And74].

Proposition 9.1.1. *Let $\mathcal{P}_N$ be a N–gon, $N \in \mathbb{Z}_{>0}$. Then*

$$\Phi_{\mathcal{P}_N}(q) \;=\; h_N(q). \tag{9.5}$$

The tail of the coloured Jones polynomial is multiplicative with respect to gluing polygons to Tait graphs.

K	$\Phi_K(q)$	$\Phi_{K^*}(q)$	K	$\Phi_K(q)$	$\Phi_{K^*}(q)$
3_1	$h_3(q)$	1	8_5	?	h_3
4_1	$h_2(q)$	$h_2(q)$	8_{18}	?	?
5_1	$h_5(q)$	1	9_{16}	$h_4(q)^2h_3(q)$	$h_4(q)$
5_2	$h_4(q)$	$h_3(q)$	$11a_{250}$	?	$h_3(q)^2$

Table 9.1.: Some identities for $\Phi_K(q)$.

Theorem 9.1.2 ([GV15, Lemma 1.5]). *Let $\mathcal{T}$ be the edge connected sum of a graph $\mathcal{T}_1$ and a N-gon P_N. Then*

$$\Phi_{\mathcal{T}}(q) \;=\; \Phi_{\mathcal{T}_1}(q)\Phi_{P_N}(q) \;=\; \Phi_{\mathcal{T}_1}(q)h_N(q). \tag{9.6}$$

So far, the modularity of $\Phi_K(q)$ has only been shown for examples of alternating knots. The main result in this section is a general formula for the tail of the coloured Jones polynomial for a class of arborescent links in terms of the functions $h_b(q)$.

With the construction of arborescent links from Chapter 8, we have the following main theorem.

Theorem 9.1.3. *Let $\Gamma = (\mathcal{V}, \mathcal{E}, w)$ be a reduced, alternating weighted tree with associated link K. If $0 \notin w(\mathcal{V}_-)$, then we have*

$$\Phi_K(q) \;=\; \prod_{v\in\mathcal{V}_+} h_{w(v)+e(v)}(q), \tag{9.7}$$

where $e(v)$ for $v \in \mathcal{V}$ denotes the number of edges adjacent to v.

Remark 9.1.4.
1. *Given a link K with associated weighted tree Γ, the mirror link K^* of K can be constructed by flipping the signs of the weights of Γ. Hence, if $0 \notin w(\mathcal{V}_+)$, we have*
$$\Phi_{K^*}(q) \;=\; \prod_{v\in\mathcal{V}_-} h_{-w(v)+e(v)}(q). \tag{9.8}$$
2. *The number of vertices of weight 0 can be reduced using the moves on weighted trees from Lemma 8.3.2 without changing the associated link. We note that the moves discussed there do not affect the formula in Theorem 9.1.3, because $h_2(q) = 1$.*
3. *The fact that Γ is reduced implies that K and K^* are prime and unsplittable. Otherwise the tail of the coloured Jones Polynomial can be computed from the components of K using* (7.22) *and* (7.23).

We study the Tait graph for arborescent links in Section 9.3 and prove the following proposition. In particular, in combination with Theorem 9.1.2, this implies Theorem 9.1.3.

Proposition 9.1.5. *Let Γ and K be as in Theorem 9.1.3 and assume that $|\mathcal{V}| \geq 2$. Then the Tait graph $\mathcal{T}_+$ of K consists of polygons of sizes $|w(v)| + e(v)$, $v \in \mathcal{V}_+$ glued together.*

The assumption $0 \notin w(\mathcal{V}_-)$ assures that the Tait graph $\mathcal{T}$ of K has no interior vertices and Theorem 9.1.2 is applicable.

The first knot K for which the modularity properties of $\Phi_K(q)$ are unknown is $K = 8_5$. It is easy to check that the formula from Theorem 7.4.1 does not hold for 8_5. There exists a representation of $\Phi_{8_5}(q)$ as a 2–fold sum which makes numerical experiments efficient. Based on asymptotic experiments, we give strong asymptotic evidence in Section 9.4 that $\Phi_{8_5}(q)$ cannot be written as a product of the (partial) theta functions $h_b(q)$. This leads to the question whether the criterion $0 \notin w(V_-)$ in Theorem 9.1.3 classifies arborescent links K for which $\Phi_K(q)$ can be written as a product of (partial) theta functions.

The first knot that is not arborescent is $K = 8_{18}$ and there is no known theta-series representation for $\Phi_K(q)$ or $\Phi_{K^*}(q)$. We discuss this knot in subsection 9.4.2.

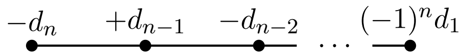
Moreover, we remark that the q-series occurring as the tail of the coloured Jones polynomial also occur as characters of VOAs [BKM19, BM17, HK03, Kan23, Kan24].

9.2. Examples and corollaries

We give examples and consequences of Theorem 9.1.3.

9.2.1. 2–bridge knots

A *2–bridge knot* can be constructed from a tree given by



for $d_1, \dots, d_n \in \mathbb{Z}_{>0}$. Then K is given in Conway's notation by the sequence $[d_1\, d_2\, \dots\, d_n]$. In Conway's terminology, 2–bridge knots are called rational.

We denote the vertices by their weights d_i and partition them into two parts $\{d_i : i \text{ odd}\}$ and $\{d_i : i \text{ even}\}$. We have $v(e_i) = 1$ if $i = 1, n$ and $v(e_i) = 2$ otherwise. With the vector

$$b \;=\; (d_i + e(v_i))_i \;=\; d + (1,\, 2,\, 2, \dots,\, 2,\, 1)^T, \tag{9.9}$$

Theorem 9.1.3 immediately implies the following result.

Corollary 9.2.1. *If K is a 2–bridge knot as above, then*

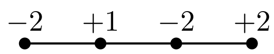
$$\Phi_K(q) \;=\; \prod_{i \text{ even}} h_{b_i}(q), \qquad \Phi_{K^*}(q) \;=\; \prod_{i \text{ odd}} h_{b_i}(q). \tag{9.10}$$

Example 9.2.2. 1. We have seen in Example 8.1.2 that the knot $K = 5_2$ is constructed from the weighted tree given in (8.1). Hence, we have

$$\Phi_K(q) \;=\; h_4(q), \qquad \Phi_{K^*}(q) \;=\; h_3(q). \tag{9.11}$$

This also follows from the Tait graphs of $K = 5_2$ given in Example 7.4.2.

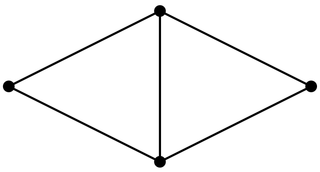
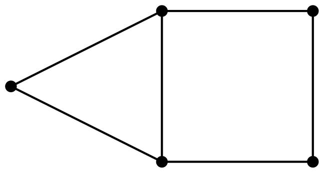
2. Let $K = 7_6 = [2\,2\,1\,2]$. An associated graph is given by



and thus, we have

$$\begin{aligned} \Phi_K(q) \;&=\; h_{2+1}(q) h_{1+2}(q) \;=\; h_3^2(q), \\ \Phi_{K^*}(q) \;&=\; h_{2+2}(q) h_{2+1}(q) \;=\; h_3(q) h_4(q). \end{aligned} \tag{9.12}$$

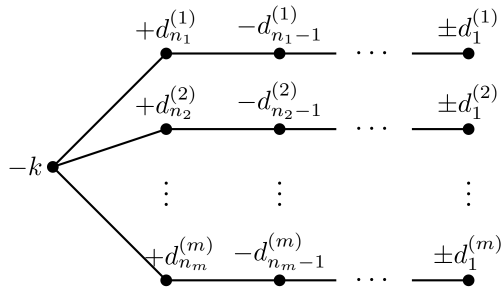
The reduced Tait graphs $\mathcal{T}_\pm$ of K are given by

$\mathcal{T}_+ \; =$  $\mathcal{T}_- \; =$ 

Since the Tait graph $\mathcal{T}_+$, resp. $\mathcal{T}_-$, consists of two edge-connected triangle and a square, resp. of a triangle and a square, (9.12) also follows from Theorem 9.1.2.

9.2.2. Montesinos links

A *Montesinos link* K can be constructed by a star-shaped tree ([AFH+21, §17.6.2]) of the form



with $m \in \mathbb{Z}_{\geq 1}$ rays of length $n_i \in \mathbb{Z}_{\geq 1}$ and $d_1^{(i)}, d_2^{(i)}, \cdots d_{n_i}^{(i)} \in \mathbb{Z}_{\geq 1}$ for $i = 1, \ldots, m$. The centre of the star has weight $-k$. The Conway notation of K is given by

$$[\mathbf{d}^{(1)}; \mathbf{d}^{(1)}; \ldots; \mathbf{d}^{(m)} +^{k}] \tag{9.13}$$

where

$$\mathbf{d}^{(i)} \;=\; [d_1^{(i)}\, d_2^{(i)}\, \ldots\, d_{n_i}^{(i)}]. \tag{9.14}$$

The number of edges for a vertex $v \in \mathcal{V}$ is given by

$$e(v) \;=\; \begin{cases} m & \text{if } v \text{ is the center,} \\ 1 & \text{if } v \text{ is a leaf,} \\ 2 & \text{otherwise} \end{cases} \tag{9.15}$$

and we define for $i = 1, \ldots, m$ the vectors

$$\mathbf{b}^{(i)} \;=\; \mathbf{d}^{(i)} + (1,\, 2,\, 2, \ldots,\, 2,\, 1)^T. \tag{9.16}$$

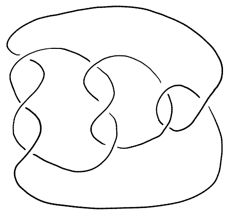
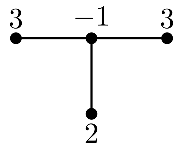
Corollary 9.2.3. *Let K be a Montesinos knot as above. Then we have*

$$\Phi_{K^*} \;=\; h_{m \pm k} \prod_{i=1}^{m} \prod_{j \not\equiv n_i \bmod 2} h_{\mathbf{b}_j^{(i)}}(q) \tag{9.17}$$

and if $k \neq 0$

$$\Phi_K \;=\; \prod_{i=1}^{m} \prod_{j \equiv n_i \bmod 2} h_{\mathbf{b}_j^{(i)}}(q). \tag{9.18}$$

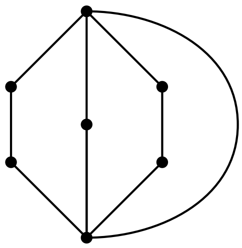
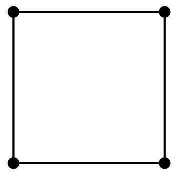
Example 9.2.4. As an example we consider the Montesinos Knot $K = 9_{16}$ with Conway notation $[3;3;2+]$. The knot and an associated graph are given by

Since no vertex has weight 0, Theorem 9.1.3 is applicable for both $\Phi_K(q)$ and $\Phi_{K^*}(q)$ and we have

$$\Phi_K(q) \;=\; h_4^2(q)h_3(q), \qquad \Phi_{K^*} \;=\; h_4(q). \tag{9.19}$$

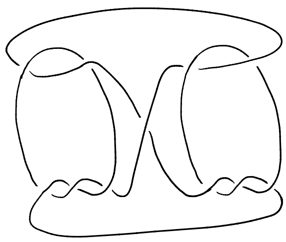
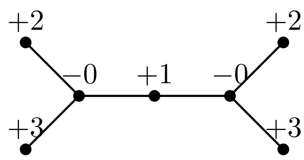
Both identities also follow from Theorem 9.1.2, since the Tait graphs of K are given by

$\mathcal{T}_+ =$  $\mathcal{T}_- =$ 

and consist of edge-connected polygons.

9.2.3. More examples

Lastly, we consider the knot $K = 11a_{250}$ with Conway notation $[(3;2)1(3;2)]$ which is not a Montesinos knot. The knot and an associated graph ([Cau82, p.38]) are given by

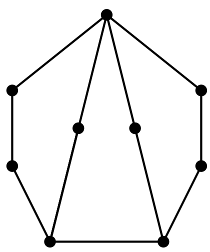
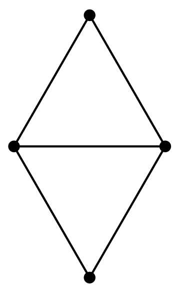



(9.20)

Hence, Theorem 9.1.3 is not applicable to K but to K^* and claims that

$$\Phi_{K^*}(q) \;=\; h_{0+3}(q)\, h_{0+3}(q) \;=\; h_3^2(q). \tag{9.21}$$

The Tait graphs of K are given by

$\mathcal{T}_+ =$  $\mathcal{T}_- =$ 

which also imply the formula for $\Phi_{K^*}(q)$.

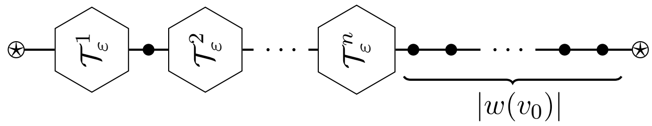
9.3. The Tait graphs for arborescent links and tangles

In this section we prove Proposition 9.1.5 and Theorem 9.1.3. Therefore, we need a recursive construction for the Tait graphs of arborescent tangles.

9.3.1. A recursive construction of the Tait graphs

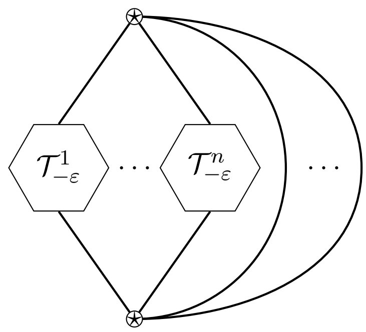
We can construct the Tait graph for arborescent tangles using the following proposition.

Proposition 9.3.1. *Let Γ be an alternating, weighted tree with root v_0 and associated tangle T. We assume that $v_0 \in \mathcal{V}_\varepsilon$ for $\varepsilon \in \{1,-1\}$ and that v_0 is connected to the subgraphs $\Gamma_1, \dots, \Gamma_n$ via the vertices $v_1, \dots, v_n \in \mathcal{V}_{-\varepsilon}$. Let $\mathcal{T}^1_\pm, \dots, \mathcal{T}^n_\pm$ denote the Tait graphs of the tangles corresponding to the tree Γ_i with root v_i. Then the Tait graph $\mathcal{T}_\varepsilon$ of Γ is given by*

$\mathcal{T}_\varepsilon =$ 

(9.22)

with $|w(v_0)| + 1$ additional vertices. Moreover, the Tait graph $\mathcal{T}_{-\varepsilon}$ of Γ is given by

$\mathcal{T}_{-\varepsilon} =$ 

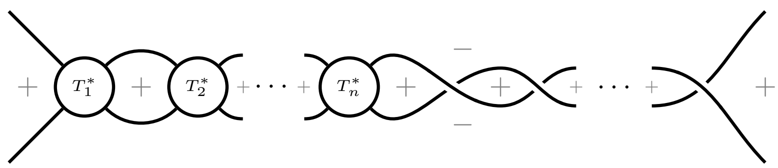
(9.23)

with $|w(v_0)|$ additional edges from the top to the bottom vertex.

We obtain the Tait graphs of a link K by picking a root v_0 among the vertices and inductively applying the previous proposition to the corresponding tangles.

If $v_0 \in \mathcal{V}_\pm$, then the Tait graph $\mathcal{T}_\pm$ of K is obtained from the $\pm$–Tait graph of the corresponding tangle by identifying the marked points. The Tait graph $\mathcal{T}_\mp$ of K is equal to the $\mp$–Tait graph of the tangle.

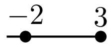
Proof. We consider the ribbon corresponding to v_0 with gluing points for $v_1, \dots, v_n$ followed by $w(v_0)$ half-twists. Without loss of generality, we assume that $v_0 \in \mathcal{V}_+$, i.e., $w(v_0) \geq 0$. If the tangles T_i correspond to Γ_i, the ribbon for v_0 has the following form. In the plumbing constructions the tangles T_i are flipped and T_i^* denotes the tangle T_i flipped along the NW and SE axis.



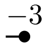
(9.24)

The $\pm$-coloured faces of the tangles of T_i^* are exactly the $\pm$-coloured faces of the tangles of T_i. Therefore, the $\pm$–Tait graph of T_i^* is equal to the $\pm$–Tait graph of T_i. Hence, the Tait graphs $\mathcal{T}_\pm$ have the claimed form. $\square$

Example 9.3.2. We consider the rooted, weighted tree

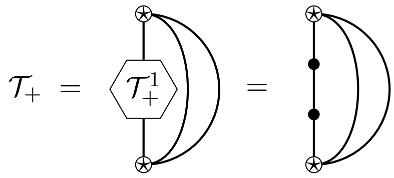


(9.25)

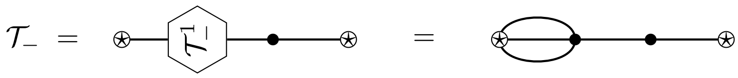
from Example 8.2.1 with root $v_0 = -2$. According to Example 8.2.2, the Tait graphs for  are given by

(9.26)

Proposition 9.3.1 implies that the Tait graphs for the tree in (9.25) are given by



(9.27)



(9.28)

in accordance with (8.6).

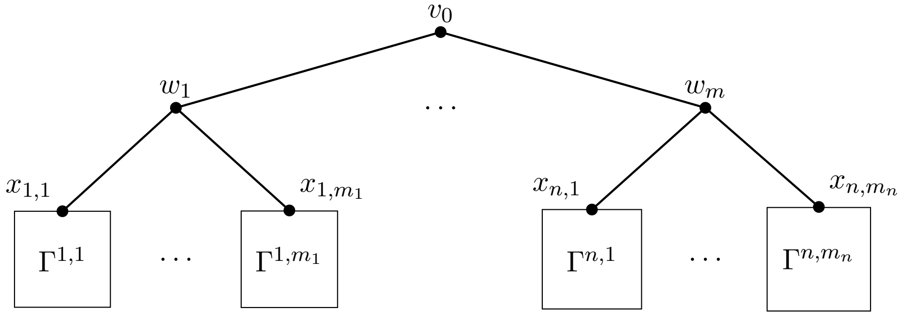
9.3.2. Proof of Theorem 9.1.3 and Proposition 9.1.5

For $|\mathcal{V}| = 1$, Theorem 9.1.3 can be checked directly. If $|\mathcal{V}| \geq 2$, Theorem 9.1.3 follows directly with Theorem 9.1.2 (see [AD11, Theorem 3.7] and [GV15, Lemma 1.5]) from Proposition 9.1.5. For the proof of Proposition 9.1.5 we need the following lemma.

Lemma 9.3.3. *Let $\Gamma = (\mathcal{V}, \mathcal{E}, w)$ be an alternating, reduced, weighted tree with $|\mathcal{V}| \geq 2$ and $0 \notin w(\mathcal{V}_-)$. For any $v_0 \in \mathcal{V}_-$ the reduced Tait graph $\mathcal{T}'_+$ of the tree Γ with root v_0 consists of edge-connected polygons of sizes $\{w(v)+e(v), v \in \mathcal{V}_+\}$ and has an edge between the two marked vertices.*

Proof. We prove the claim via induction over $|\mathcal{V}|$. First assume that $|\mathcal{V}| = 2$ with $\mathcal{V}_\pm = \{v_\pm\}$. Since by assumption Γ has no leaf of weight 0, we have $w(v_+) \neq 0$. It follows from Example 8.2.2 that $\mathcal{T}'_+$ is a polygon of size $w(v_+) + 1$.

Now assume that the claim is true for all graphs with less than $|\mathcal{V}|$ vertices. Let $v_0 \in \mathcal{V}_-$ be connected to $w_1, \ldots, w_n \in \mathcal{V}_+$ and w_i be connected to the subgraphs $\Gamma_{i,j}$ (with vertices $\mathcal{V}_{i,j}$) via the vertices $x_{i,j} \in \mathcal{V}_-$ for $j = 1, \ldots, m_i$ and $i = 1, \ldots, m$.



Let $\mathcal{T}_+^{i,j}$ be the reduced Tait graphs for $\Gamma^{i,j}$ with roots $x_{i,j}$. By assumption, $\mathcal{T}_+^{i,j}$ consists of edge-connected polygons of size $\{w(v)+e(v), v \in (\mathcal{V}_{i,j})_+\}$ and has an edge between the poles.

Applying Proposition 9.1.5 twice implies that the reduced Tait graph $\mathcal{T}'_+$ for Γ with root v_0 has the form as depicted in Figure 9.1 with an edge between the marked vertices because $w(v_0) \neq 0$.

In particular, $\mathcal{T}'_+$ consists of edge-connected polygons of sizes

$$\bigcup_{i,j}\{w(v)+e(v) : v \in (\mathcal{V}_{i,j})_+\} \cup \bigcup_{i=1}^{n}\{w(w_i)+m_i\} \;=\; \{w(v)+e(v) : v \in \mathcal{V}_+\}$$

since $m_i = e(w_i)$.

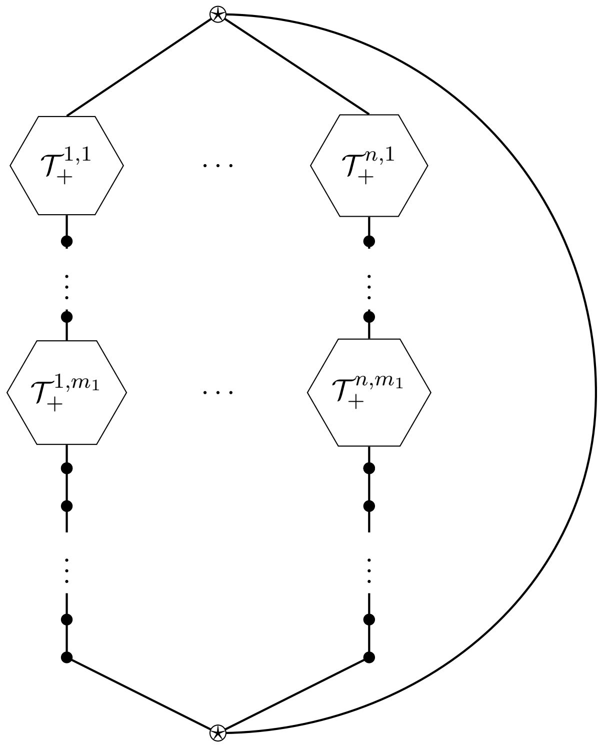


Figure 9.1.: The Tait graph $\mathcal{T}_+$

□

Proof of Proposition 9.1.5. Pick $v_0 \in \mathcal{V}_+$ and assume that v_0 is connected via $w_1, \ldots, w_n \in \mathcal{V}_-$ to the subgraphs with vertices $\mathcal{V}^i$. By Proposition 9.3.1, $\mathcal{T}'_+$ has the form as in equation (9.22) with $\pm = +$.

By assumption, Γ has no leaf of weight 0 and the previous lemma implies that $\mathcal{T}_+^i$ for $i = 1, \ldots, n$ consists of polygons of size $\{w(v)+e(v), v \in \mathcal{V}_+^i\}$ and the two poles are connected by an edge. This implies that $\mathcal{T}_+$ consists of polygons of sizes

$$\bigcup_{i=1}^{n}\{w(v)+e(v), v \in \mathcal{V}_+^i\} \;\cup\; \{w(v_0)+e(v_0)\} = \{w(v)+e(v), v \in \mathcal{V}_+\}.$$

□

9.4. Non-modularity of $\Phi_K(q)$

The condition $0 \notin w(\mathcal{V}_-)$ in Theorem 9.1.3 implies that the Tait graph $\mathcal{T}_+$ of K is the edge-connected sum of polygons.

In this case, the known formulas for the tails $\Phi_K(q)$ are applicable. However, Theorem 9.1.3 does not apply to any case where a formula for $\Phi_K(q)$ is not known, e.g., the entries with "?" in Table 9.1.

The first arborescent knot with this property is 8_5 and it is easy to check that the suggested identity $\Phi_K(q) = h_4(q)^2 h_3(q)$ from Theorem 9.1.3 does not hold. We give numerical evidence that the tail Φ_{8_5} is not modular and cannot be written as a product of $h_b(q)$'s. This leads to the question of whether the criterion $0 \in w(\mathcal{V}_-)$ classifies all arborescent knots such that $\Phi_K(q)$ can be written as a product of h_b's.

Theorem 7.4.1 gives a representation for Φ_{8_5} as an 8–fold sum and it is possible to reduce the sum to a 2–fold-sum (see Theorem 9.4.1 below). Using this representation, the tail can be computed efficiently. Then we can compare the asymptotics of $\Phi_{8_5}(q)$ and $h_b(q)$ as $q \to 1$ as we did for ordinary Nahm sums in Part I of this thesis.

It is easy to prove that

$$h_b(e^{-h}) \;\sim\; \begin{cases} e^{-\pi^2/2bh}\sqrt{\frac{2\pi}{bh}}\cos(2\pi(\frac{1}{4}-\frac{1}{2b})) & \text{if } b \text{ is odd,} \\ \frac{2}{b} & \text{if } b \text{ is even} \end{cases} \tag{9.29}$$

as $h \to 0$ on any ray in the right half-plane. Hence, if a function f with asymptotics $f(e^{-h}) \sim e^{V/h} h^k \beta$ as $h \to 0$ for some $V, \beta \in \mathbb{C}$ and $k \in \frac{1}{2}\mathbb{Z}$ can be written as a product of $h_b(q)$'s, the asymptotics in (9.29) would imply that $V \in \pi^2\mathbb{Q}$. Even though this condition cannot be verified numerically, numerical computation can suggest if $V \in \pi^2\mathbb{Q}$ is likely. For example, if numerics suggest that V is not even real, then $V \notin \pi^2\mathbb{Q}$ is very likely. Note that the asymptotics would also exclude that Φ_K can be written as a linear combination of products of theta functions with coefficients in $q^\alpha\mathbb{C}[q^{\pm}]$, $\alpha \in \mathbb{Q}$.

9.4.1. Examples: Pretzel knots

A pretzel knot $P(n_1, \ldots, n_l)$ is an arborescent knot associated with a star-shaped tree with l rays of length 1 where the leafs have weight $n_1, \ldots, n_l$. The tail of the coloured Jones polynomial for pretzel knots of the form $K = P(2k+1, 2, 2u+1)$ for $k, i \in \mathbb{Z}_{\geq 1}$ can be computed explicitly.

Theorem 9.4.1 ([EH17]). *The tail of the coloured Jones polynomials for a pretzel knot of the form $K = P(2k+1, 2, 2u+1)$ for $k, u \in \mathbb{Z}_{\geq 1}$ is given by*

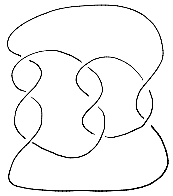
$$\Phi_K(q) = (q;q)_\infty^2 \sum_{l_1 \geq 0} \cdots \sum_{l_k \geq 0} \sum_{p_1 \geq 0} \cdots \sum_{p_u \geq 0} \frac{q^{L_1^2+\ldots+L_k^2+L_1+\cdots L_k}}{(q)_{l_1}\cdots(q)_{l_{k-1}}(q)_{l_k}^2} \frac{q^{P_1^2+\ldots+P_u^2+P_1+\cdots P_u}}{(q)_{p_1}\cdots(q)_{p_{u-1}}(q)_{p_u}^2}(q)_{l_k+p_u}, \tag{9.30}$$

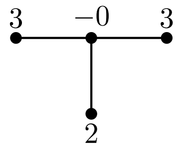
where

$$L_j \;=\; l_j + \cdots + l_k, \qquad P_j \;=\; p_j + \cdots + p_u. \tag{9.31}$$

The first case is given by $k = u = 1$ corresponding to the knot $K = 8_5$ as in (8.3).

Example 9.4.2. We consider the tail $\Phi_K(q)$ for $K = 8_5$ with Conway notation $[3;2;3]$. A knot diagram and an associated weighted tree are given by





(9.32)

see also (8.3) and (8.7). According to Theorem 9.4.1, we have

$$\begin{aligned}\Phi_{8_5}(q) &= (q)_\infty^2 \sum_{a,b\geq 0} \frac{q^{a^2+a+b^2+b}}{(q)_a^2 (q)_b^2}(q)_{a+b} \\ &= 1 - 2q + q^2 - 2q^4 + 3q^5 - 3q^8 + q^9 + \mathrm{O}(q^{10}).\end{aligned} \tag{9.33}$$

As $h \searrow 0$ and $q = e^{-h} \nearrow 1$, numerics suggest that we have

$$\Phi_{8_5}(e^{-h})(q)_\infty^{-2} \sim e^{V_1/h}\sqrt{\frac{1}{h\pi\beta}} \tag{9.34}$$

where with $X \approx 0.5436890$ a root of $x^3 + x^2 + x - 1$

$$\begin{aligned}V_1 &= -4\,\mathrm{Li}_2(X) + \mathrm{Li}_2(X^2) - 2\log(X)^2 + \frac{\pi^2}{6} \\ &= -1.352936859\cdots \\ \beta &= 3 + 4X + X^2.\end{aligned} \tag{9.35}$$

This can presumably be proven similar to the asymptotics of usual Nahm sums in Section 4. Computing V_1 to a higher precision, it seems very unlikely that $V_1 \in \pi^2\mathbb{Q}$ is true. This suggests that $q^c\Phi_{8_5}(q)$ for any $c \in \mathbb{Q}$ is not a modular form/function and cannot be written as a product of the functions $h_b(q)$, $b \in \mathbb{Z}_{\geq 1}$. Computing the asymptotics as $h \to 0$ on a fixed ray in the right half-plane as in Theorem 4.3.1 with $\arg h = .45\pi$ is even more convincing: Numerics suggest that

$$\Phi_{8_5}(e^{-h})(q)_\infty^{-2} \sim e^{V_2/h}\sqrt{\frac{h}{\pi\beta}} \tag{9.36}$$

where with $X \approx -0.7718445 - 1.115143i$ a root of $x^3 + x^2 + x - 1$

$$\begin{aligned}V_2 &= -4\,\mathrm{Li}_2(X) + \mathrm{Li}_2(X^2) - 2\log(X)^2 - 4\pi i\log(X) + \frac{\pi^2}{2} \\ &= -10.83807\ldots + 3.177293\ldots i, \\ \beta &= 12 + 16X + 4X^2\end{aligned} \tag{9.37}$$

and $V_2 \notin \pi^2\mathbb{Q}$ because it is not even real.

If we replace the weight 0 by -1 in the weighted tree for 8_5, we obtain the knot 9_{16}. We have seen in Example 9.2.4 that both $\Phi_{9_{16}}(q)$ and $\Phi_{9_{16}^*}(q)$ can be written as products of h_b's.

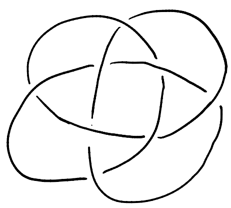
Similar computations have been performed for pretzel knots of the form $K = P(2k + 1, 2, 2u + 1)$ with $k, u \leq 5$ using equation (9.30). These computations suggest that for all these pretzel knots the asymptotics is given by $\Phi_K(e^{-h}) \sim ah^k e^{V/h}$ as $h \nearrow 0$ for some $a \in \mathbb{C}$, $k \in \frac{1}{2}\mathbb{Z}$, and $\mathrm{V} \notin \pi^2\mathbb{Q}$. Therefore, we suspect that $\Phi_K(q)$, cannot be written as a product of the functions $h_b(q)$. It would be desirable to consider more examples where Theorem 9.1.3 is not applicable to obtain more evidence for an answer to the following question.

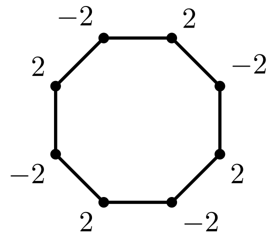
Question 9.4.3. *Does Theorem 9.1.3 classify arborescent, alternating links K such that $\Phi_K(q)$ is a product of (partial) theta functions? In other words, is $\Phi_K(q)$ a product of (partial) theta functions if and only if there exists an associated weighted tree with $0 \notin w(\mathcal{V}_-)$?*

9.4.2. Non-arborescent knots

So far we have only discussed the modularity of the tail of the coloured Jones polynomial for arborescent knots. The first knot that is not arborescent is $K = 8_{18}$ and we discuss the tail of the coloured Jones polynomial for K next.

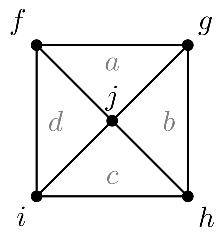
Even though $K = 8_{18}$ cannot be constructed from a weighted tree, K can be constructed from the weighted graph ([Cau82, Con70]) given by an octagon





(9.38)

Following Theorem 7.4.1, the tail of the coloured Jones polynomial for 8_{18} can be written as an explicit 8-fold sum. The Tait graphs of $K = 8_{18}$ are given by

$\mathcal{T}_\pm =$ 

(9.39)

With respect to the variables $a, b, c, d, e, f, g, h, i, j$, we define

$$Q = \left(\begin{array}{ccccc|ccccc} 3 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 4 & 1 & 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right), \qquad b = \left(\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ \hline 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right). \tag{9.40}$$

We set $e=0$, as well as $f=0$, and require that all of the following are non-negative

$$\begin{array}{ccccc} e+f=0 & f+d=d & f+a=a & g+b & h+c \\ e+g=g & j+d & g+a & h+b & i+c \\ e+h=h & i+d & j+a & j+b & j+c \\ e+i=i & & & & \end{array} \tag{9.41}$$

Then, using $1/(q)_n = 0$ for $n<0$, we have

$$\begin{aligned}
&\Phi_K(q) \\
&= (q)_\infty^8 \sum_{\substack{s\in\mathbb{Z}^{10} \\ s_5=s_6=0}} (-1)^{2b^Ts} \frac{q^{s^TQs/2+b^Ts}}{\prod_{s_i\in s_j}(q)_{s_i+s_j}} \\
&= (q)_\infty^8 \sum_{a,b,c,d,g,h,i,j\in\mathbb{Z}} (-1)^{a+b+c+d} \frac{q^{3a^2/2+ag+aj+a/2+3b^2/2+bg+bh+bj+b/2+3c^2/2+ch+ci+cj+c/2}}{(q)_g(q)_h(q)_i(q)_d(q)_a(q)_{j+d}(q)_{i+d}} \\
&\quad\times \frac{q^{3d^2/2+di+dj+d/2+gh+gj+g+hi+hj+h+ij+i+j}}{(q)_{g+a}(q)_{j+a}(q)_{g+b}(q)_{h+b}(q)_{j+b}(q)_{h+c}(q)_{i+c}(q)_{j+c}} \\
&= 1-4q+2q^2+9q^3-5q^4-8q^5-14q^6+10q^7+21q^8+14q^9+O(q^{10})
\end{aligned}$$

where we identified $s=(a,b,c,d,e=0,f=0,g,h,i,j)$.

Given the first terms of $\Phi_{8_{10}}(q)$, one easily checks that a result similar to Theorem 9.1.3 is not true, i.e., $\Phi_{8_{18}}(q) \neq h_3(q)^4$. In addition to that, one can check numerically that $\Phi_{8_{18}}(q)$ is not a product of $h_b(q)$'s for $b\leq 10$. A computable representation of $\Phi_{8_{18}}(q)$ would be desirable in order to make any precise conjectures about the modularity of $\Phi_{8_{18}}(q)$.

9.4.3. Modularity and the Bloch-group

The q–series representation of $\Phi_K(q)$ in equation (7.27) can be seen as a generalisation of Nahm sums as discussed in Part I of this thesis. Since the modularity of (generalised) Nahm sums is believed to be related to elements in the Bloch group, we expect something similar to hold here. As in the context of Nahm sums, the conjectured non-modularity of $\Phi_{8_5}(q)$ in Example 9.4.2 relies on the non-vanishing of $[X]\in\mathcal{B}(\mathbb{C})$.

Recall that the Volume conjecture (Conjecture 7.2.8) provides a different connection between the (quantum) modularity of the coloured Jones polynomial for a knot K and certain elements in the Bloch group. Recently, Garoufalidis-Zagier [GZ23, GZ24] refined the Volume conjecture providing more evidence for the quantum modularity related to the coloured Jones polynomial. However, the element in the Bloch group $2[X]$ appearing in Example 9.4.2 seems to be different to the one Garoufalidis and Zagier consider. For example, the knot $K=4_1$ is hyperbolic and thus the q-series studied by Garoufalidis-Zagier is not modular in a classical sense. However, 4_1 is a 2–bridge knot with Conway notation [2 2] and we have seen in Section 9.2 that $q^{\frac{1}{24}}\Phi_K(q) = q^{\frac{1}{24}}h_3(q) = \eta(\tau)$ is modular, where $q=e^{2\pi i\tau}$.

Even though the asymptotics in the Volume conjecture look similar to the asymptotics for $\Phi_K(q)$ (e.g., in (9.34)), they are quite different. See also the discussion in [Gar18,

§7]. In the limit in (7.17) for the Volume conjecture we consider simultaneously $N \to \infty$ and $q = e^{2\pi i/N} \to 1$ on the unit disc. In the context of the tail of the coloured Jones polynomial, we first take the limit of the normalised coloured Jones polynomial to obtain its tail before considering $q \to 1$ in the unit disk in (9.34).

Part III.

Partitions and q-series

10. Partitions

10.1. Introduction

In this part of the thesis, we will use q-series to study classes of partitions and related questions. The main focus lies on a conjecture of Andrews [And86] concerning the sign pattern of coefficients of a function from Ramanujan's "lost" notebook. This part is mainly based on the preprint [FMRS23] that is joint work with Amanda Folsom, Joshua Males, and Larry Rolen. The main contribution of the author of this thesis to the preprint is the proof of Theorem 4.3.1 presented in subsection 11.1. We summarise the proofs of Theorem 10.2.3 and Theorem 10.2.4 in Section 11.2, following [FMRS23].

A *partition* of $n \in \mathbb{Z}_{\geq 0}$ is a sequence of non-decreasing integers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l$ for $l \in \mathbb{Z}_{\geq 1}$ with $\sum_{i=1}^{l} \lambda_i = n$. For example, all partitions of 4 are given by

$$4 \;=\; 3+1 \;=\; 2+2 \;=\; 2+1+1 \;=\; 1+1+1+1. \tag{10.1}$$

Denote by $p(n)$ the number of partitions of n. For example, (10.1) gives $p(4) = 5$. We define the *partition function*

$$P(q) \;=\; \sum_{n\geq 0} p(n)q^n \;=\; 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots + 3972999029388q^{200} + \cdots . \tag{10.2}$$

The advantage of considering the series $P(q)$ is that the properties of the function $q \mapsto P(q)$ reveal information about the coefficients $p(n)$. The function $P(q)$ can be written as

$$P(q) \;=\; \sum_{l\geq 0} \sum_{\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_l} q^{\lambda_1+\lambda_2+\cdots+\lambda_l}. \tag{10.3}$$

If we expand $\lambda_1 = k_1 \geq 0$, $\lambda_i = \lambda_{i-1} + k_2$ for some $k_i \geq 0$ and $i = 1, \ldots, l$, we obtain by computing each sum as a geometric sum

$$P(q) \;=\; \sum_{n\geq 0} \sum_{k_1\geq 0} \cdots \sum_{k_n\geq 0} q^{k_1+2k_2+\cdots+nk_n} \;=\; \prod_{i\geq 1} \frac{1}{(1-q^i)} \;=\; (q;q)_\infty^{-1}, \tag{10.4}$$

with the q-Pochhammer symbol defined in (2.40). Recall that $q^{-1/24}(q;q)_\infty = \eta(\eta)$ for $q = e^{2\pi i\tau}$ is the Dedekind eta-function, see (2.53).

Hardy-Ramanujan developed the circle method to prove the asymptotic expansion

$$p(n) \;\sim\; \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}} \qquad \text{as } n \to \infty \tag{10.5}$$

using the asymptotics of $P(q)$ for q near a root of unity. Rademacher [Rad43] refined the circle method and made use of the modularity of $q^{\frac{1}{24}}P(q) = \eta(\tau)^{-1}$, where $q = e^{2\pi i\tau}$ in order to find an exact formula for the coefficients $p(n)$ of $P(q)$.

We will briefly describe the heuristics behind the circle method, since we will use an adapted version in Section 10.2 below.

10.1.1. The circle method

Let $f(q) = \sum_{n\geq 0} a(n)q^n$ be a holomorphic function with coefficients $a(n) \in \mathbb{Z}$. The idea behind the circle method is to use Cauchy's integral formula to write

$$a(n) \;=\; \frac{1}{2\pi}\int_C \frac{f(q)}{q^{n+1}}dq, \tag{10.6}$$

where C is a circle going counterclockwise around 0 near the unit disc. We will split up the integral into *major arcs*, meaning the most significant parts of the integral, and *minor arcs*, containing the rest of the integral. Assume that the most significant contribution in the integral comes from the part where q is near a root of unity $\zeta \in \mathbb{C}$ with

$$f(\zeta e^{-h}) \;\sim\; e^{V/h}h^k\lambda, \qquad \text{as } h \to 0 \tag{10.7}$$

for some $V, k, \lambda \in \mathbb{R}$ with $V > 0$. Then for some small $\varepsilon > 0$, the integral in (10.6) can heuristically be estimated by the part where q is near ζ, suggesting

$$a(n) \;\sim\; \frac{1}{2\pi}\int_C \frac{f(q)}{q^{n+1}}dq \;\sim\; \frac{\lambda}{2\pi\zeta^n n^{(k+1)/2}}\int_{(-\varepsilon,\varepsilon)} e^{\sqrt{n}(V/z+z)}z^k dz. \tag{10.8}$$

Bounding the error terms of course takes more care. After expanding the integral around the saddle point of the integral $z = \sqrt{V}$, we expect

$$a(n) \;\sim\; \frac{\lambda}{2\sqrt{\pi}\zeta^n}\frac{V^{\frac{k}{2}+\frac{1}{4}}}{n^{\frac{k}{2}+\frac{3}{4}}}\exp(2\sqrt{Vn}) \tag{10.9}$$

as $n \to \infty$.

For example, for $P(q)$ as in (10.2) we know that the major contribution in the integral (10.6) comes from q near $\zeta = 1$, see Lemma 4.4.2, with

$$P(e^{-h}) \;\sim\; e^{\pi^2/6h}\frac{h^{1/2}}{\sqrt{2\pi}} \tag{10.10}$$

and the heuristics discussed above with $V = \frac{\pi^2}{6}$, $k = \frac{1}{2}$, and $\lambda = \frac{1}{\sqrt{2\pi}}$ suggest the asymptotic formula (10.5) found by Hardy-Ramanujan.

10.1.2. Nahm sums and generating functions for partitions

In the one-dimensional case, Nahm sums, studied in Part I of this thesis, are the generating functions for classes of partitions. For simplicity, we assume that $A \in \mathbb{Z}_{\geq 1}$ and $b \in \frac{A}{2} + \mathbb{Z}$, meaning that the quadratic form $\frac{1}{2}An^2 + bn$ has no denominator.

Proposition 10.1.1. *For $n \in \mathbb{Z}_{\geq 0}$, the summand $b_n(q) = \frac{q^{\frac{1}{2}An^2+bn}}{(q)_n}$ is the generating function for partitions $1 \leq \lambda_1 \leq \ldots \leq \lambda_n$ with*

$$\lambda_1 \geq \frac{A}{2} + b, \qquad \lambda_i \geq \lambda_{i-1} + A \text{ for } i = 2, \ldots, n. \tag{10.11}$$

Moreover, the Nahm sum $f_{A,b}(q)$ as defined in (3.7) is the generating function for partitions satisfying (10.11) for some $n \in \mathbb{Z}_{\geq 0}$.

Proof. If we expand each factor in the q-Pochhammer symbol as a geometric series, we obtain

$$b_n(q) \;=\; \frac{q^{\frac{1}{2}An^2+bn}}{(q)_n} \;=\; q^{\frac{1}{2}An^2+bn}\sum_{l_1\geq 0}\cdots\sum_{l_n\geq 0} q^{1l_1+2l_2+\cdots+nl_n}. \tag{10.12}$$

By substituting $k_j = l_1+\cdots+l_j$ for $j=1,\ldots,n$, we obtain with $n^2=\sum_{i=1}^n(2i-1)$

$$\begin{aligned} b_n(q) \;&=\; q^{\frac{1}{2}An^2+bn}\sum_{0\leq k_1\leq\cdots\leq k_n} q^{k_1+\ldots+k_n} \\ &=\; \sum_{0\leq k_1\leq\cdots\leq k_n} q^{\sum_{i=1}^n\left(\frac{A}{2}(2i-1)+b_i+k_i\right)}. \end{aligned} \tag{10.13}$$

If we set

$$\lambda_i \;=\; \left(\frac{1}{2}A(2i-1)+b_i+k_i\right), \quad i=1,\ldots,n, \tag{10.14}$$

then $\lambda_1\leq\cdots\leq\lambda_n$ is a partition satisfying (10.11). In other words,

$$b_n(q) = \sum_{\lambda} q^{\sum_{i=1}^n\lambda_i}, \tag{10.15}$$

where the sum is over partitions λ satisfying (10.11). This proves the first claim. Summing over all $n\geq 0$ gives the second claim. ☐

For example, for $A=2$ and $b=0$, the Rogers-Ramanujan function $G(q)=f_{2,0}(q)$, cf. (3.1), is the generating function for partitions whose parts differ by at least 2.

The ideas from Proposition 10.1.1 can be used to examine more classes of partitions and their properties. For example, in a project with Bringmann, Man, and Rolen [BMRS23], we examine the parity bias of partitions with distinct parts following questions posed by Kim, Kim, and Lovejoy [KKL20]. If $d_{\mathrm{odd}}(n)$ denotes the number of partitions of n into distinct parts with more odd than even parts, then the generating function of $d_{\mathrm{odd}}(n)$ is given by

$$\sum_{n\geq 0} d_{\mathrm{odd}}(n)q^n \;=\; \sum_{n_1>n_2\geq 0}\frac{q^{n_1(n_1+1)+n_2^2}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}}. \tag{10.16}$$

Moreover, let $d_{\mathrm{even}}(n)$ be the number of partitions of n into distinct parts with more even than odd parts. Using the asymptotics of (10.16) for q near roots of unity, we show in [KKL20] that

$$d_{\mathrm{odd}}(n)-d_{\mathrm{even}}(n) \;\sim\; \frac{e^{\pi\sqrt{n/3}}}{8\sqrt{6}n} \tag{10.17}$$

as $n\to\infty$. In particular, $d_{\mathrm{odd}}(n)>d_{\mathrm{even}}(n)$ for n large enough.

10.2. A sign pattern conjecture of Andrews

In the study of Ramanujan's "lost" notebook, Andrews found several q-series with interesting behaviour. For instance, Andrews [And86] notices that, while the coefficients of most q-series either "tend to infinity in absolute value or are bounded", the coefficients of the function

$$\begin{aligned}\sigma(q) &:= \sum_{n\geq 0} \frac{q^{\frac{1}{2}n(n+1)}}{(-q;q)_n} = \sum_{n\geq 0} S(n)q^n \\ &= 1 + q - q^2 + 2q^3 - 2q^4 + q^5 + q^7 - 2q^8 + \cdots \\ &\quad + 2q^{4962} + 4q^{4963} + 8q^{4967} - 2q^{4968} - 4q^{4980} + \cdots .\end{aligned}$$

behave differently: Even though they seem to grow very slowly, it seems that we have $\limsup |S(n)| = \infty$. This behaviour was later explained by Andrews-Dyson-Hickerson [ADH88] by relating the coefficients $S(n)$ to solutions of quadratic equations and establishing a relation to indefinite theta functions. In [Coh88], Cohen constructs a *Maass waveform* from $\sigma(q)$, relating its coefficients to the arithmetic of $\mathbb{Q}(\sqrt{6})$.

Besides $\sigma(q)$, Andrews also considers the function

$$\begin{aligned}v_1(q) &:= \sum_{n\geq 0} \frac{q^{\frac{1}{2}n(n+1)}}{(-q^2;q^2)_n} = \sum_{n\geq 0} V_1(n)q^n \\ &= 1 + q + q^6 - q^7 - q^8 + q^9 + q^{10} + \cdots \\ &\quad + 762q^{700} + 8365q^{701} - 273q^{702} - 8550q^{703} \\ &\quad - 224q^{704} + 8716q^{705} + 761q^{706} - 8832q^{707} + \cdots\end{aligned} \tag{10.18}$$

from Ramanujan's "lost" notebook [And84]. Andrews notes that, even though the "growth of $|V_1(n)|$ is not very smooth", there appears to be "great sign regularity in $V_1(n)$". He also makes some conjectures.

Conjecture 10.2.1 ([And86, Conjectures 3-6]).

1. *$|V_1(n)| \to \infty$ as $n \to \infty$ for almost all $n \in \mathbb{Z}_{\geq 0}$.*
2. *For almost all $n \in \mathbb{Z}_{n\geq 0}$, four consecutive coefficients $V_1(n)$, $V_1(n+1)$, $V_1(n+2)$, and $V_1(n+3)$ consist of two positive and two negative numbers.*
3. *For $n \geq 5$ there is an infinite sequence $N_5 = 293, N_6 = 410, N_7 = 545, N_8 = 702, \ldots,$ $N_n \geq 10n^2, \ldots$ such that $V_1(N_n)$, $V_1(N_n+1)$ and $V_1(N_n+2)$ all have the same sign.*
4. *The numbers $|V_1(N_n)|$, $|V_1(N_n+1)|$, $|V_1(N_n+2)|$ contain a local minimum of the sequence $(V_1(j))_{j\geq 0}$.*

We remark that part 1 of Andrews's conjecture originally reads "$|V_1(n)| \to \infty$ as $n \to \infty$". Based on computations and theoretic examinations, we have modified the conjecture slightly.

The coefficients $V_1(n)$ for $n \leq 1000$ are plotted in Figure 10.1. From Figure 10.1, we see that the asymptotics of $V_1(n)$ appear to depend on $n \pmod 4$. Moreover, the sequence can be divided into 4 reoccurring sections with the following patterns for $\operatorname{sign}(V_1(n))$, where $n \equiv n_0 \mod 4$.

n_0	0	1	2	3
Section 1	+	−	−	+
Section 2	+	+	−	−
Section 3	−	+	+	−
Section 4	−	−	+	+

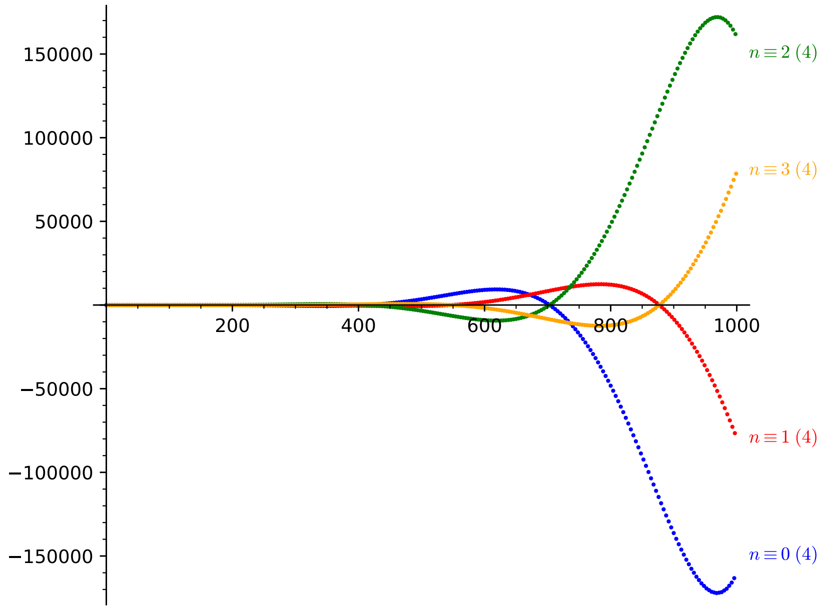


Figure 10.1.: $V_1(n)$ for $n \leq 1000$

For example, in Figure 10.1 and (10.18), we see the sign pattern $+--+$ between $n = 546$ and $n = 702$ (Section 1), the sign pattern $++--$ between $n = 703$ and $n = 877$ (Section 2), etc. We ultimately establish the sign regularity of $V_1(n)$ in Theorem 10.2.3 below.

The coefficients $V_1(n)$ have a combinatorial interpretation: A partition is called *odd-even* if the parity of the parts alternates with the smallest part odd. The *rank* of a partition is defined as the difference between the largest part and the number of parts. For example, the rank of the partition $9+4+1$ is $9-3=6$. It is easy to show that the rank of an odd-even partition is always even. The coefficients $V_1(n)$ count the difference between the number of odd-even partitions of n with rank congruent to $0 \pmod 4$ and congruent to $2 \pmod 4$.

We explain Andrews's conjecture following the circle method described above: In Theorem 10.2.2, we give the asymptotics of $v_1(q)$ as q approaches a root on unity. This leads to the asymptotics of the coefficients $V_1(n)$ in Theorem 10.2.3.

Similar to the asymptotics of ordinary Nahm sums (Theorem 4.3.1), a solution of the modified Nahm equation and their dilogarithm value appear in the asymptotics of $v_1(q)$. For $v_1(q)$, the modified Nahm equation is given by

$$(1-Q)^2 \;=\; -Q \tag{10.19}$$

with solutions $Q = e^{\pm\pi i/3} = \frac{1\pm\sqrt{3}i}{2}$. In contrast to the case of ordinary Nahm sums, there is no solution in $(0,1)$. The fact that the solutions consist of a pair of complex conjugated complex numbers will produce oscillation in the asymptotics of $v_1(q)$. This will lead to oscillation in the asymptotics of the coefficients $V_1(n)$, explaining the sign pattern of $V_1(n)$.

Similar to the asymptotics in Theorem 4.3.1, the different asymptotic contributions in $v_1(q)$, corresponding to $Q = e(\pm 1/6)$, will be visible when considering the asymptotics $q = \pm i e^{-z}$, $z \to 0$ on different rays in the right half-plane. Depending on $\arg z$, one contribution will be exponentially big while the other one will be exponentially smaller.

Recall the convention $e(x) = e^{2\pi i x}$ for $x \in \mathbb{C}$.

Theorem 10.2.2. *Let $\zeta = e(\alpha) \in \mathbb{C}$ be a root of unity of order $m \in \mathbb{N}_0$.*

1. *If $4 \nmid m$, then $v_1(\zeta e^{-z}) = O(1)$ as $z \to 0$ along any ray in the right half-plane.*
2. *If $4|m$, then as $z \to 0$, on a ray in the right half-plane with $0 \neq |\arg z| < \frac{\pi}{2}$, we have*
$$\begin{aligned} v_1(\zeta e^{-z}) \;&=\; \exp\left(\frac{16V}{zm^2}\right)\sqrt{\frac{2\pi i}{z}}\gamma^+_{(\alpha)}(z)(1+O(z^L)) \\ &+\; \exp\left(\frac{-16V}{zm^2}\right)\sqrt{\frac{2\pi i}{-z}}\gamma^-_{(\alpha)}(z)(1+O(z^L)) \end{aligned} \tag{10.20}$$
for all $L > 0$, where V is given in terms of the Bloch-Wigner dilogarithm D*, see* (2.12)*, by*
$$V \;=\; \mathrm{D}(e(1/6))\frac{i}{8} \;=\; 0.1268677\cdots i, \tag{10.21}$$
and the power series $\gamma^\pm_{(\alpha)}(z) \in \mathbb{C}[[z]]$ are defined in subsection 11.1.3.
3. *In particular, for $\zeta = \pm i$, the power series are explicitly defined in* (11.33) *and* (11.42) *and the first terms are given by*
$$\gamma^+_{(1/4)}(z) \;=\; \overline{\gamma^-_{(3/4)}(z)} \;=\; \gamma^+\left(1+\left(\frac{1}{3}-\frac{77}{216}\sqrt{3}\right)iz+\left(-\frac{89449}{31104}+\frac{647}{648}\sqrt{3}\right)z^2+O(z^3)\right),$$
$$\gamma^-_{(1/4)}(z) \;=\; \overline{\gamma^+_{(3/4)}(z)} \;=\; \gamma^-\left(1+\left(\frac{1}{3}+\frac{77}{216}\sqrt{3}\right)iz+\left(-\frac{89449}{31104}-\frac{647}{648}\sqrt{3}\right)z^2+O(z^3)\right),$$
where
$$\begin{aligned} \gamma^+ \;&:=\; \gamma^+_{(1/4)}(0) \;=\; \gamma^-_{(3/4)}(0) \;=\; \frac{1}{2\sqrt[4]{3(2-\sqrt{3})}} \;=\; 0.5280518\cdots, \\ \gamma^- \;&:=\; \gamma^-_{(1/4)}(0) \;=\; \gamma^+_{(3/4)}(0) \;=\; \frac{1}{2\sqrt[4]{3(2+\sqrt{3})}} \;=\; 0.2733397\cdots. \end{aligned}$$

The proof of the previous theorem is given in Section 11.1. By applying an adapted version of the circle method to $v_1(q)$ with major arc corresponding to q near the roots of unity $\pm i$, we will establish the asymptotics of $V_1(n)$ as $n \to \infty$.

Theorem 10.2.3. *As $n \to \infty$, the coefficients $V_1(n)$ are asymptotically equal to*
$$\begin{aligned} &(-1)^{\lfloor \frac{n}{2} \rfloor}\,\frac{e^{\sqrt{2|V|n}}}{\sqrt{n}}(\gamma^+ + (-1)^n\gamma^-)\left(\cos(\sqrt{2|V|n}) - (-1)^n\sin(\sqrt{2|V|n})\right)\left(1+O\left(n^{-\frac{1}{2}}\right)\right) \\ &+O\left(n^{-\frac{1}{2}}e^{\sqrt{\frac{|V|n}{2}}}\right). \end{aligned} \tag{10.22}$$

We give the proof in Section 11.2 below. The sequence $V_1(n)e^{-\sqrt{2|V|n}}\sqrt{n}$ is plotted in Figure 10.2. Using these results we are able to prove parts of Andrews's conjecture.

Theorem 10.2.4. *Part 1 and 2 in Andrews's Conjectures 10.2.1 are true.*

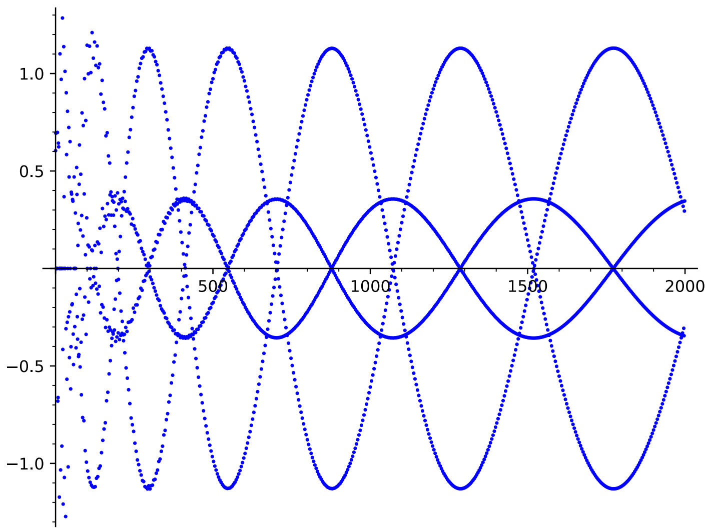


Figure 10.2.: $V_1(n)e^{-\sqrt{2|V|n}}\sqrt{n}$ for $0 \le n \le 2000$

We will prove the theorem after discussing and explaining Andrews's Conjectures 10.2.1. For this, we use the asymptotics of $V_1(n)$ from Theorem 10.2.3 and note that

$$(-1)^{\lfloor \frac{n}{2} \rfloor} = \begin{cases} 1 & \text{if } n = 0, 1 \bmod 4, \\ -1 & \text{if } n = 2, 3 \bmod 4. \end{cases} \tag{10.23}$$

Hence, as long as the term

$$\mathcal{F}_{\pm}(n) := \left(\cos(\sqrt{2|V|n}) \pm \sin(\sqrt{2|V|n})\right) \tag{10.24}$$

in (10.22) is large enough, four consecutive terms of the sequence $V_1(n)$ will consist of two positive and two negative numbers, explaining part 2 of Andrews's conjecture 10.2.1. When the functions $\mathcal{F}_{\pm}(x)$, $x \in \mathbb{R}$, become 0, the sign pattern will change. Note that the zeros of the functions $\mathcal{F}_{\pm}$ are given by

$$\vartheta_j = \left(j + \frac{1}{2}\right)^2 \frac{\pi^2}{8|V|}, \qquad j \in \mathbb{Z}_{\ge 0}. \tag{10.25}$$

Therefore, the sign pattern changes whenever $n = \lfloor \vartheta_j \rfloor$ for some $j \in \mathbb{Z}_{\ge 0}$. If the sign pattern changes, three consecutive numbers in the sequence will have the same sign. Note that these three numbers can already start before the sign pattern changes. Hence, the numbers N_j from Conjecture 10.2.1 should fulfil $|N_j - \lfloor \vartheta_j \rfloor| \le 3$. The first values for N_j and $\lfloor \vartheta_j \rfloor$ are listed in the following table.

j	5	6	7	8	9	10	11	12	13	14
N_j	293	410	545	702	877	1072	1285	1518	1771	2044
$\lfloor \vartheta_j \rfloor$	294	410	546	702	877	1072	1286	1519	1772	2044

A natural related question is when the coefficients $V_1(n)$ vanish. After checking the first five million coefficients of $v_1(q)$, it appears that $V_1(n)$ vanishes only finitely often and exactly if

$$n \in \{2, 3, 4, 5, 11, 13, 15, 17, 19, 21, 25, 29, 31, 39, 47, 58, 60, 62, 64, 101, 111, 123, 129, 198\}.$$

It is reasonable to expect that the leading term in (10.22) becomes the most significant term as $n \to \infty$. This leading term is zero whenever $n = \vartheta_j$ (cf. (10.25)), which can only happen for $n \in \mathbb{Z}$ if $|V| \in \pi^2\mathbb{Q}$. This suggests that the vanishing of $V_1(n)$ is related to the rationality of $|V|/\pi^2$. Values of the dilogarithm are known to be related to zeta-values, see e.g., [Zag07]. Here, we have

$$|V| \;=\; \frac{\operatorname{Im} \mathrm{D}(e^{\pi i/3})}{8} \;=\; \frac{9\sqrt{3}\zeta_{\mathbb{Q}(\sqrt{-3})}(2)}{16\pi^2} \tag{10.26}$$

where $\zeta_K(s)$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ is the Dedekind zeta function associated with the number field K (see e.g., [Neu99]). The rationality of zeta-values is an important question in number theory. For totally real fields K, the Siegel-Klingen theorem ([Kli62, Sie80, Sie69]) implies that $\zeta_K(2k) \in |\operatorname{disc}(K)|^{-\frac{1}{2}}\pi^{2k[K:\mathbb{Q}]}\,\mathbb{Q}$ for $k \in \mathbb{Z}_{\geq 1}$. Hecke [Hec20] already gave the result for real quadratic fields K. However, for imaginary quadratic fields it seems currently not possible to make any general statements about the rationality of $\zeta_K(2)$.

Zagier [Zag86] generalised Euler's theorem ($\zeta_{\mathbb{Q}}(2) = \frac{\pi^2}{6}$) to number fields K by proving a formula that relates $\zeta_K(2)$ to powers of π, $\sqrt{\operatorname{disk}(K)}$, and integrals of the form

$$A(x) \;=\; \int_0^x \frac{1}{1+t^2}\log\frac{4}{1+t^2}\,dt. \tag{10.27}$$

Nevertheless, Zagier's result does not imply anything about the rationality of $\frac{|V|}{\pi^2}$. Numerical computations suggest that $|V| \notin \pi^2\mathbb{Q}$, which is in agreement with the numerical observation that $V_1(n)$ vanishes only finitely often.

Proof of Theorem 10.2.4. Following [FMRS23], we prove Theorem 10.2.4, which states that the followind parts of Andrews's conjceture 10.2.1 are true.

1. $|V_1(n)| \to \infty$ as $n \to \infty$ for almost all $n \in \mathbb{Z}_{\geq 0}$.
2. For almost all $n \in \mathbb{Z}_{\geq 0}$, four consecutive coefficients $V_1(n)$, $V_1(n+1)$, $V_1(n+2)$, and $V_1(n+3)$ consist of two positive and two negative numbers.

We denote the main term in (10.22) by

$$M(n) \;=\; (-1)^{\lfloor \frac{n}{2} \rfloor}\,\frac{e^{\sqrt{2|V|n}}}{\sqrt{n}}(\gamma^+ + (-1)^n\gamma^-)\left(\cos(\sqrt{2|V|n}) - (-1)^n\sin(\sqrt{2|V|n})\right) \tag{10.28}$$

and collect the rest in an error term, denoted by $E(n)$. We need to prove that the main term $M(n)$ is bigger than the error term $E(n)$ for almost all $n \in \mathbb{Z}_{\geq 0}$. The inequality $M(n) > E(n)$ occurs when $\mathcal{F}_{\pm}(n) > e^{\sqrt{|V|n/2}}$, i.e.,

$$|n - \vartheta_j| > e^{-\sqrt{|V|n/2} + \varepsilon\sqrt{n}} \tag{10.29}$$

for all $j \in \mathbb{Z}_{\geq 0}$ and some $\varepsilon > 0$. A result of Schoißengeier [Sch81] states that for the *discrepancy* of the sequence $\frac{1}{2\pi}\sqrt{2|V|n}$ for $n = 1, \ldots, N$ we have

$$\sup_{0 \leq c \leq d \leq 1}\left|\frac{|\{\frac{1}{2\pi}\sqrt{2|V|n} \bmod 1 :\; n = 1\ldots, N\} \cap [c,d]|}{N} - (d-c)\right| \;\ll\; O(N^{-\frac{1}{2}}). \tag{10.30}$$

It follows that the number of points $n \in \mathbb{Z}_{\geq 0}$ not satisfying $M(n) > E(n)$ is bounded below by $1 - O(n^{-1/2})$. In other words, the numbers $n \in \mathbb{Z}_{\geq 0}$ for which $M(n) > E(n)$

is at least $1 - O(n^{-1/2})$, which converges to 1 as $n \to \infty$. This implies $|V_1(n)| \to \infty$ for almost all n, proving part 1 in Andrews's conjecture 10.2.1. Moreover, this implies that for almost all n, four consecutive numbers $V_1(n)$,$V_1(n+1)$,$V_1(n+2)$, and $V_1(n+3)$ will consist of two positive and two negative numbers. This proves part 2 in Andrews's conjecture 10.2.1. $\square$

11. Proofs

11.1. Proof of Theorem 10.2.2: The asymptotics of $v_1(q)$

We will prove each part in Theorem 10.2.2 separately. We will provide details for the proof for $\zeta = i$ and only sketch the steps for the general case.

11.1.1. Proof of Theorem 10.2.2, (1)

We begin by showing that at any root of unity with order not divisible by 4, $v_1(q)$ converges.

Lemma 11.1.1. *Let $\zeta = e(\frac{l}{m})$ with $\gcd(l,m) = 1$ and $4 \nmid m$. Then Theorem 10.2.2, Part (1) is true and we have as $z \to 0$ on a ray in the right half-plane*

$$v_1(\zeta e^{-z}) \;\to\; v_1(\zeta) \;=\; 2\sum_{s=0}^{m-1} \frac{\zeta^{s(s+1)}}{(-\zeta^2;\zeta^2)_s}. \tag{11.1}$$

Proof. For ζ as above, we compute

$$v_1(\zeta) \;=\; \sum_{n\geq 0} \frac{\zeta^{n(n+1)/2}}{(-\zeta^2;\zeta^2)_n} \;=\; \sum_{s=0}^{m-1}\sum_{l\geq 0} \frac{\zeta^{s+mk(s+mk+1)/2}}{(-\zeta^2;\zeta^2)_{s+mk}}, \tag{11.2}$$

where we substituted $n = s+mk$. From $(-\zeta^2;\zeta)_{s+mk} = 2(-\zeta^2;\zeta^2)_s$ we deduce

$$v_1(\zeta) \;=\; \sum_{s=0}^{m-1}\sum_{l\geq 0} \frac{\zeta^{s+mk(s+mk+1)/2}}{2^k(-\zeta^2;\zeta^2)_s} \;= 2\sum_{s=0}^{m-1} \frac{\zeta^{s(s+1)/2}}{(-\zeta^2;\zeta^2)_s} \tag{11.3}$$

which is finite. Moreover, as $z \to 0$ on a ray in the right half-plane, z will eventually be in a Stolz sector and Abels theorem (see e.g., [Ahl78, §2.5]) implies that $v_1(\zeta e^{-z}) \to v_1(\zeta)$. $\square$

11.1.2. Proof of Theorem 10.2.2, (3)

Throughout, we assume that $\varphi \in \mathbb{C}$ with $|\varphi| = 1$ and $0 \neq |\arg(\varphi)| < \frac{\pi}{2}$ and write $z = \varphi h$. We will present the case $q = ie^{-z} \to i$ for $z = \varphi h \to 0$ on a fixed ray in the right half-plane in detail. The case $q \to -i$ is analogous, and we omit the proof for brevity.

We split up the sum defining $v_1(q)$ depending on $n \pmod 2$, i.e., consider separately

$$\begin{aligned} v_1^{[0]}(q) \;&=\; \sum_{n\geq 0 \text{ even}} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n} \;=\; \frac{1}{(-q^2;q^2)_\infty} \sum_{n\geq 0 \text{ even}} (-i)^{n/2} e^{-zn(n+1)/2}\, (-e^{-2nz}q^2;q^2)_\infty, \\ v_1^{[1]}(q) \;&=\; \sum_{n\geq 0 \text{ odd}} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n} \;=\; \frac{1}{(-q^2;q^2)_\infty} \sum_{n\geq 0 \text{ odd}} i^{(n+1)/2} e^{-zn(n+1)/2}\, (e^{-2nz}q^2;q^2)_\infty, \end{aligned} \tag{11.4}$$

as we have

$$i^{n(n+1)/2} \;=\; \begin{cases} (-i)^{n/2}, & \text{if } n \text{ is even,} \\ i^{(n+1)/2}, & \text{if } n \text{ is odd.} \end{cases}$$

To state our next result, recall that for $n \geq 0$ the q-Pochhammer symbol satisfies the classical formula

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

meaning that we may extend the definition of the q-Pochhammer symbol to $-n$ by defining

$$(a;q)_{-n} \coloneqq \frac{(a;q)_\infty}{(aq^{-n};q)_\infty} = \frac{1}{(aq^{-n};q)_n}.$$

Then the q-Pochhammer symbol satisfies

$$(a;q)_{-n} \;=\; q^{n(n-1)/2}\frac{(-q/a)^n}{(q/a;q)_n}. \tag{11.5}$$

We will prove the following proposition which implies (3) in Theorem 10.2.2.

Proposition 11.1.2. *As $z \to 0$ in the right half-plane on a ray with $\arg z \neq 0$, we have with $q = ie^{-z}$*

$$v_1^{[0]}(q) \;= e^{-\frac{V}{z}}\sqrt{\frac{2\pi i}{-z}}\;\gamma^-_{(1/4)}(z)(1+O(z^L)) + \phi^{[0]}_{(1/4)}(z)(1+O(z^L)), \tag{11.6}$$

$$v_1^{[1]}(q) \;= e^{\frac{V}{z}}\sqrt{\frac{2\pi i}{z}}\;\gamma^+_{(1/4)}(z)(1+O(z^L)) + \phi^{[1]}_{(1/4)}(z)(1+O(z^L)) \tag{11.7}$$

for all $L > 0$, where $\gamma^{\pm}_{(1/4)}(z) \in \mathbb{C}[[z]]$ are defined in (11.33), resp. (11.42) and power series $\phi^{[0]}_{(1/4)}(z), \phi^{[1]}_{(1/4)}(z) \in \mathbb{C}[[z]]$ with

$$\begin{aligned} \phi^{[0]}_{(1/4)}(z) \;&=\; \sum_{\substack{n<0: \\ n\equiv 0 \bmod 2}} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n} \;=\; -4i\,z - 48\,z^2 + \frac{2878}{3}i\,z^3 + 26704\,z^4 + O(z^5), \\ \phi^{[1]}_{(1/4)}(z) \;&=\; \sum_{\substack{n<0: \\ n\equiv 1 \bmod 2}} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n} \;=\; 2 + 8i\,z - 96\,z^2 - \frac{5708}{3}i\,z^3 + 52640\,z^4 + O(z^5). \end{aligned} \tag{11.8}$$

Watson's contour integral

Using Watson's contour integral ([GR11, 4.2], [Wat10]), we establish the following integral representation for $v_1^{[0]}(q)$ and $v_1^{[1]}(q)$.

Lemma 11.1.3. *For $q = ie^{-z}$ with $\mathrm{Re}(z) > 0$, we have*

$$v_1^{[0]}(q) \;=\; \frac{-1}{2i}\frac{1}{(-q^2;q^2)_\infty}\int_{L_\infty} e^{\pi is/4}e^{-zs(s+1)/2}(-e^{-2sz}q^2;q^2)_\infty\frac{1}{2\sin(\pi s/2)}ds, \tag{11.9}$$

$$v_1^{[1]}(q) \;=\; \frac{e^{\pi i3/4}}{2}\frac{1}{(-q^2;q^2)_\infty}\int_{L_\infty} e^{-\pi is/4}e^{-zs(s+1)/2}(e^{-2sz}q^2;q^2)_\infty\frac{1}{2\cos(\pi s/2)}ds, \tag{11.10}$$

where L_∞ is the contour depicted in Figure 11.1 as $R \to \infty$ for some small $\varepsilon > 0$.

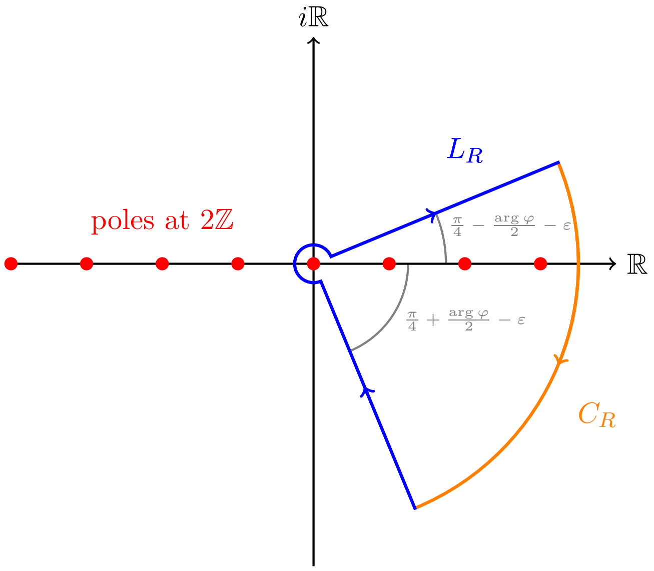


Figure 11.1.: Contours L_R and C_R

Proof. We will prove the statement for $v_1^{[0]}(q)$ in detail. The proof for $v_1^{[1]}(q)$ follows analogously.

As mentioned above, we write $z = \varphi h \in \mathbb{C}$ where $h \in \mathbb{R}_{>0}$ and $\varphi \in \mathbb{C}$ with $|\varphi| = 1$ and $|\arg\varphi| < \frac{\pi}{2}$. The function $\frac{1}{\sin(\pi s/2)}$ has poles at $s \in 2\mathbb{Z}$ with residues $(-1)^{s/2}\,\frac{2}{\pi}$. Hence, with $(-i)^{s/2}(-1)^{s/2} = i^{s/2} = e^{\pi i s/4}$ for $s \in 2\mathbb{Z}$, we obtain by Cauchy's theorem - if the subsequent integrals are convergent - using the contours from Figure 11.1

$$\begin{aligned}
&\frac{-1}{2i}\,\frac{1}{(-q^2;q^2)_\infty}\lim_{R\to\infty}\int_{L_R+C_R} e^{\pi i s/4}e^{-zs(s+1)/2}(-e^{-2sz}q^2;q^2)_\infty\frac{1}{2\sin(\pi s/2)}ds \\
&= \frac{1}{(-q^2;q^2)_\infty}\sum_{n\geq 0 \text{ even}}\operatorname*{Res}_{s=n}\left((-i)^{s/2}e^{-zs(s+1)/2}(-e^{-2sz}q^2;q^2)_\infty\frac{\pi(-1)^{s/2}}{2\sin(\pi s/2)}\right) \\
&= \frac{1}{(-q^2;q^2)_\infty}\sum_{n\geq 0 \text{ even}}(-i)^{n/2}e^{-zn(n+1)/2}\,(-e^{-2nz}q^2;q^2)_\infty \\
&= v_1^{[0]}(q).
\end{aligned} \tag{11.11}$$

It remains to prove the following 2 claims.

1. The integral over L_∞ converges.

2. The integral over the arc C_R vanishes as $R \to \infty$.

Before proving (1) and (2), we make some initial observations. If we parameterise L_R and C_R away from the indention around 0 by $s = re^{i\theta}$ with $0 < r \leq R$ and $\theta \in \left[-\frac{\pi}{4} - \frac{\arg\varphi}{2} + \varepsilon, \frac{\pi}{4} - \frac{\arg\varphi}{2} - \varepsilon\right]$, then we have $-zs^2 = -hr^2e^{i(\arg\varphi+2\theta)}$ because we have $\arg\varphi + 2\theta \in [-\frac{\pi}{2} + 2\varepsilon, \frac{\pi}{2} - 2\varepsilon]$ for $\varepsilon > 0$, i.e., $\mathrm{Re}(-zs^2) < 0$. Similarly, one checks $\mathrm{Re}(-zs) < 0$.

In particular, the Pochhammer symbol can be uniformly bounded by

$$\begin{aligned}
\left|(-e^{-2sz}q^2;q^2)_\infty\right| &\leq \prod_{j\geq 1} 1+|e^{-2sz}||q^{2j}| \\
&= \prod_{j\geq 1} 1+|e^{-2\mathrm{Re}(sz)}||e^{-2j\mathrm{Re}(z)}| \\
&< \prod_{j\geq 1} 1+|e^{-2j\mathrm{Re}(z)}|,
\end{aligned} \tag{11.12}$$

since $-\mathrm{Re}(sz) < 0$. Hence, we have

$$\begin{aligned}
&\left|\int_{L'_R+C_R} e^{\pi is/4}e^{-zs(s+1)/2}(-e^{-2sz}q^2;q^2)_\infty \frac{1}{2\sin(\pi s/2)}\right| \\
&\quad \leq \prod_{j\geq 1}(1+|e^{-2j\mathrm{Re}(z)}|)\int_{L'_R+C_R} e^{\mathrm{Re}(\pi is/4-zs(s+1)/2)}\left|\frac{1}{2\sin(\pi s/2)}\right|\,ds,
\end{aligned} \tag{11.13}$$

where L'_R denotes the part of L_R away from the indention around 0.

As $\mathrm{Im}|s| \to \infty$, we have for all $L \in \mathbb{N}$

$$|\sin(\pi s/2)| = \frac{1}{2}e^{\pi|\mathrm{Im}(s)|/2}(1+o(|s|^{-L})$$

and with $s = re^{i\theta}$ we compute

$$\begin{aligned}
&\mathrm{Re}\left(\frac{\pi is}{4} - \frac{zs(s+1)}{2}\right) - \frac{\pi|\mathrm{Im}(s)|}{2} \\
&= \mathrm{Re}\left(\frac{\pi ire^{i\theta}}{4} - \frac{he^{i\arg\varphi}r^2e^{2i\theta}}{2} - \frac{he^{i\arg\varphi}re^{i\theta}}{2}\right) - \frac{\pi|\mathrm{Im}(s)|}{2} \\
&= -\frac{\pi r\sin(\theta)}{4} + \mathrm{Re}\left(-\frac{hr^2e^{i(\arg\varphi+2\theta)}}{2} - \frac{hre^{i(\arg\varphi+\theta)}}{2}\right) - \frac{\pi|\mathrm{Im}(s)|}{2} \\
&= -\frac{\pi r\sin(\theta)}{4} - \frac{r^2h^2\cos(\arg\varphi+2\theta)}{2} - \frac{rh\cos(\arg\varphi+\theta)}{2} - \frac{r\pi|\sin\theta|}{2} \\
&= -\frac{r^2h^2\cos(\arg\varphi+2\theta)}{2} - r\left(\frac{\pi\sin(\theta)}{4} + \frac{h\cos(\arg\varphi+\theta)}{2} - \frac{\pi|\sin\theta|}{2}\right).
\end{aligned} \tag{11.14}$$

Hence, as $R \to \infty$, the exponent in the integrand in (11.13) is eventually negative, since $\arg\varphi + 2\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and thus $\cos(\arg\varphi + 2\theta) > \delta > 0$ for some δ.
More precisely, we have for some constant $M > 0$, uniformly in θ,

$$\begin{aligned}
&-\frac{r^2h^2\cos(\arg\varphi+2\theta)}{2} - r\left(\frac{\pi\sin(\theta)}{4} + \frac{h\cos(\arg\varphi+\theta)}{2} - \frac{\pi|\sin\theta|}{2}\right) \\
&\quad < -\frac{r^2h^2\delta}{2} + r\left(\frac{\pi}{4} + \frac{h}{2} - \frac{\pi}{2}\right) < -Mr^2
\end{aligned} \tag{11.15}$$

for R and r large enough.

Therefore, it is sufficient to prove both claims for the integral

$$\int e^{-Mr^2}\,ds. \tag{11.16}$$

Claim (1): The integral over L_∞ converges.
We consider the integral along the contour $\{re^{i\theta_\pm}, r \in \mathbb{R}_{>0}\}$ with $\theta_\pm = \pm\frac{\pi}{4} - \frac{\arg\varphi}{2}$. By the discussion above, the integral is

$$O\left(\int_0^R e^{-Mr^2}\, dr\right), \tag{11.17}$$

which converges as $R \to \infty$.
Claim (2): The integral over the arc C_R vanishes as $R \to \infty$.
Similarly, we wee that the integral over C_R is eventually bounded by a constant times

$$\int_{\left(-\frac{\pi}{4}-\frac{\arg\varphi}{2}, \frac{\pi}{4}-\frac{\arg\varphi}{2}\right)} e^{-MR^2}\, d\theta \to 0 \tag{11.18}$$

as $R \to \infty$. □

Sum over even integers

We will prove the statement for $v_1^{[0]}$ in detail. As the proof for $v_1^{[1]}$ follows mutatis mutandis, we will only sketch it in subsection 11.1.2.

Proof of (11.6). We use the integral representation from Lemma 11.1.3 with $q = ie^{-z}$ and substitute $s = iv/z$ to obtain

$$v_1^{[0]}(q) = \frac{-1}{2z(-q^2;q^2)_\infty}\int_{-izL_\infty} e^{-\pi v/4z} e^{v^2/2z - iv/2}(-e^{-2iv}q^2;q^2)_\infty \frac{1}{\sin\left(\frac{\pi iv}{2z}\right)} dv, \tag{11.19}$$

where the contour $-izL_\infty$ is depicted in Figure 11.2.

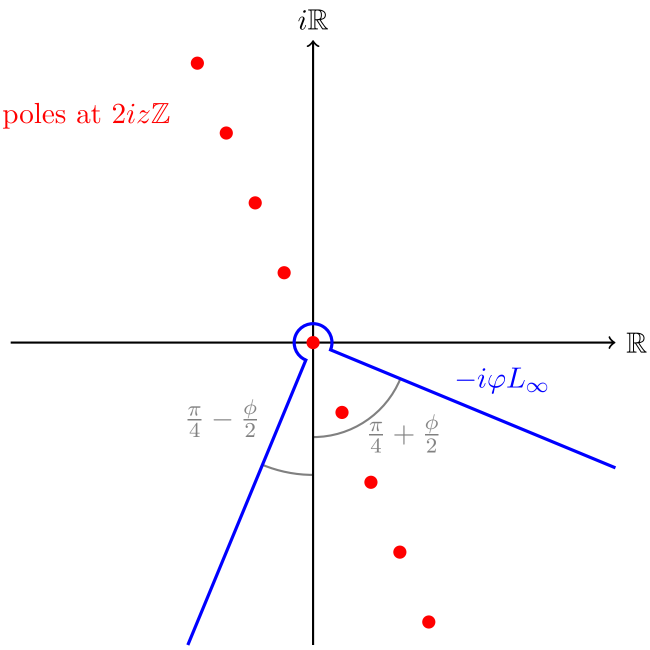


Figure 11.2.: The Contour $-i\varphi L_\infty$

We consider the integral representation (11.19) of $v_1^{[0]}$ and change the contour $-izL_\infty$ to a contour $\mathcal{S}$ with fixed minimum distance from 0 and passing through $-\frac{\pi}{12}$. The poles of the integrand in (11.19) are at $v \in i2z\mathbb{Z}$, and as $z \to 0$ they accumulate at 0. Hence, if we integrate along the contour $\mathcal{S}$, all poles at $i2z\mathbb{Z}_{<0}$ eventually get shifted to the other side of the contour (cf. Figure 11.3). In other words the integral (11.19) can be written as

$$\begin{aligned}&\frac{-1}{2z(-q^2;q^2)_\infty}\int_{\mathcal{S}} e^{-\pi v/4z}e^{v^2/2z-iv/2}(-e^{-2iv}q^2;q^2)_\infty\frac{1}{\sin\left(\frac{\pi iv}{2z}\right)}dv\\&+\frac{-1}{2z(-q^2;q^2)_\infty}\sum_{\substack{n<0:\\|2zn|<d_0}}\operatorname*{Res}_{v=-2izn}\left(e^{-\pi v/4z}e^{v^2/2z-iv/2}(-e^{-2iv}q^2;q^2)_\infty\frac{1}{\sin\left(\frac{\pi iv}{2z}\right)}\right)\end{aligned}\tag{11.20}$$

for some $d_0 > 0$, which is the distance from 0 to where the contour crosses the line of poles. The residue of $\frac{1}{\sin\left(\frac{\pi iv}{2z}\right)}$ at $v = -2izn$ is given by $-(-1)^n 2z$ and thus

$$\operatorname*{Res}_{v=-2izn}\left(e^{-\pi v/4z}e^{v^2/2z-iv/2}(-e^{-2iv}q^2;q^2)_\infty\frac{1}{\sin\left(\frac{\pi iv}{2z}\right)}\right)=(-i)^n e^{-(2n^2-n)z}(-e^{-4nz}q^2;q^2)_\infty.\tag{11.21}$$

As $z \to 0$ the residues can be collected in

$$\begin{aligned}\phi^{[0]}_{(1/4)}(z) &:= \frac{-1}{2z(-q^2;q^2)_\infty}\sum_{n<0}\operatorname*{Res}_{v=-2izn}\left(e^{-\pi v/4z}e^{v^2/2z-iv/2}(-e^{-2iv}q^2;q^2)_\infty\frac{1}{\sin\left(\frac{\pi iv}{2z}\right)}\right)\\&= \frac{1}{(-q^2;q^2)_\infty}\sum_{n<0}(-i)^n e^{-(2n^2-n)z}(-e^{-4nz}q^2;q^2)_\infty.\end{aligned}\tag{11.22}$$

Using the q-Pochhammer symbol for negative indices, this implies with $q = ie^{-z}$

$$\begin{aligned}\phi^{[0]}_{(1/4)}(z) &= \sum_{\substack{n<0:\\n\equiv 0 \bmod 2}}\frac{q^{n(n+1)/2}}{(-q^2;q^2)_n}\\&= \sum_{\substack{l>0:\\l\equiv 0 \bmod 2}} q^{-l(l-1)/2}(-1;q^2)_l\\&= 2\sum_{\substack{m>0:\\m\equiv 1 \bmod 2}} q^{-m(m+1)/2}(-q^2;q^2)_m.\end{aligned}\tag{11.23}$$

We have

$$(-q^2;q^2)_m = \prod_{j=1}^{m} 1+(-1)^j e^{-2jz} \in z^{\lceil m/2\rceil}\mathbb{C}[[z]]\tag{11.24}$$

since $1+(-1)^j e^{-2jz} \in z\mathbb{C}[[z]]$ for $j = 1,\dots,m$ odd. Hence, $\phi^{[0]}_{(1/4)}(z) \in \mathbb{C}[[z]]$ and the first terms are given by

$$\begin{aligned}\phi^{[0]}_{(1/4)}(z) &= -4i\,z - 48\,z^2 + \frac{2878}{3}i\,z^3 + 26704\,z^4 - \frac{28574401}{30}i\,z^5 + \frac{5643616}{5}\,z^6\\&\quad - \frac{106567268641}{1260}i\,z^7 + \frac{2071812944}{105}\,z^8 - \frac{289882093403521}{90720}i\,z^9 + O(z^{10}).\end{aligned}\tag{11.25}$$

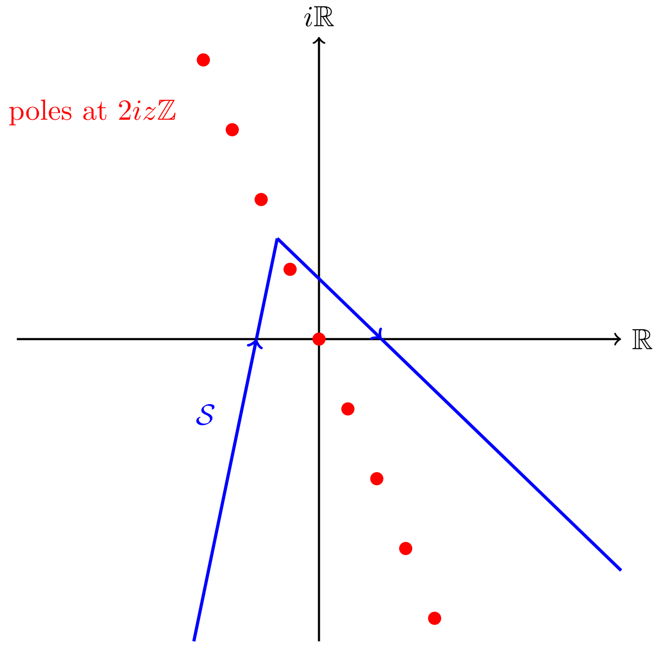


Figure 11.3.: A contour $\mathcal{S}$ (before applying Lemma 4.4.2).

We apply Lemmas 4.4.2 and 4.4.1 to the integrands in (11.20), to obtain that $v_1^{[0]}(q)$ is asymptotically equal to

$$\frac{ie^{\pi^2/48z-z/12}}{2z}\int_{\mathcal{S}}\operatorname{sign}(\operatorname{Re}(v/\varphi))\exp\left(\frac{-\operatorname{Li}_2^{\varphi}(e^{-4iv})+4v^2-2\pi v-4\operatorname{sign}(\operatorname{Re}(v/\varphi))\pi v}{8z}\right)$$
$$\exp\left(-\frac{iv}{2}+\frac{\operatorname{Li}_1^{\varphi}(-e^{-2iv})}{2}+\psi_{-1}(4z;-e^{-2iv})\right)dv\ +\ \phi^{[0]}_{(1/4)}(z) \tag{11.26}$$

which can be rewritten with

$$\begin{aligned} f(v) &= -\frac{\operatorname{Li}_2^{\varphi}(e^{-4iv})}{8}+\frac{v^2}{2}-\frac{\pi v}{4}-\operatorname{sign}(\operatorname{Re}(v/\varphi))\frac{\pi v}{2}, \\ g(z;v) &= \operatorname{sign}(\operatorname{Re}(v/\varphi))\exp\left(-\frac{iv}{2}+\frac{\operatorname{Li}_1^{\varphi}(-e^{-2iv})}{2}+\psi_{-1}(4z;-e^{-2iv})\right), \end{aligned} \tag{11.27}$$

as

$$v_1^{[0]}(q) = \frac{i}{2z}e^{\pi^2/48z-z/12}\int_{\mathcal{S}}e^{f(v)/z}g(z;v)dv\ (1+O(z^L))\ +\ \phi^{[0]}_{(1/4)}(z)(1+O(z^L)) \tag{11.28}$$

for all $L>0$. Note that both f and g are holomorphic functions on the domain

$$\mathbb{C}\smallsetminus\left(\{\varphi i\mathbb{R}_{<0}\}\bigcup_{0\neq n\in\mathbb{Z}}\{n+\varphi i\mathbb{R}_{>0}\}\right) \tag{11.29}$$

as $\operatorname{Li}_2^{\varphi}(z)$ jumps by $2\pi i\log z$ when z crosses the cut on $\operatorname{Re}(z/\varphi)=0, \operatorname{Re}(z)>0$. Moreover, $e^{\frac{\operatorname{Li}_1(-e^{-2iv})}{2}}$ changes the sign when v crosses the line $\operatorname{Re}(z/\varphi)=0,\ \operatorname{Re}(z)>0$. The contour and the branch cuts of f are plotted in Figure 11.4.

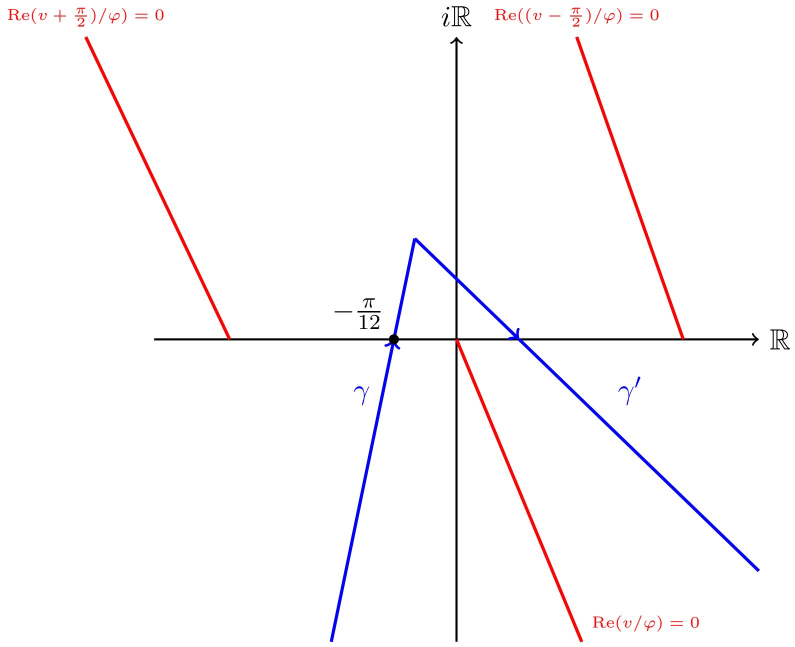


Figure 11.4.: The contours $\mathcal{S}$, parameterised by γ, γ' (after applying Lemma 4.4.2).

We compute

$$\begin{aligned} f'(v) &= -\frac{i}{2}\log(1-e^{-4iv})+v-\frac{\pi}{4}-\mathrm{sign}(\mathrm{Re}(v/\varphi))\frac{\pi}{2} \\ f''(v) &= \frac{1+e^{-4iv}}{1-e^{-4iv}} \end{aligned}$$

and thus the critical points v_0 of f satisfy

$$(1-e^{-4iv_0})^2 = -e^{-4iv_0}, \tag{11.30}$$

in other words $e^{-4iv_0}=e(\pm\frac{1}{6})$. Using this along with the condition $f'(v)=0$ at saddle points, one may classify all saddle points of f.

Using Cauchy's theorem, we choose an explicit contour (see [FMRS23] for details) passing through $v=-\frac{\pi}{12}$. By a specific parametrisation of the contour, in combination with the maximum modulus principle, one can show that all cases can be estimated using the case for $\varphi=i$. This implies that the chosen contour passing through $v_0=-\frac{\pi}{12}$ is indeed a stationary point.

We write $Q:=e^{-4iv_0}=e(1/6)=\frac{1+\sqrt{3}i}{2}$. If we parameterise $\mathcal{S}$ as $v=-\frac{\pi}{12}+is\sqrt{z}$ in a small neighbourhood around v_0 we obtain that the contribution corresponding to the stationary point $v_0=-\frac{\pi}{12}$ is given by

$$\begin{aligned} &\frac{-1}{2\sqrt{z}}e^{\pi^2/48z+f(v_0)/z-z/12}\int\exp\Big(-f''(-\tfrac{\pi}{12})\frac{s^2}{2}+\sum_{l\geq 3}\frac{f^{(l)}(Q)}{l!}(is)^l z^{l/2-1}\Big)g\Big(-\frac{\pi}{12}+is\sqrt{z};z\Big)ds \\ &=\sqrt{\frac{2\pi i}{-z}}e^{\pi^2/48z+f(v_0)/z}\,\gamma^-_{(1/4)}(z), \end{aligned} \tag{11.31}$$

where

$$\frac{\pi^2}{48}+f(v_0) = \frac{\pi^2}{48}-\frac{\mathrm{Li}_2(e^{-4iv_0})}{8}+\frac{v_0^2}{2}+\frac{\pi v_0}{4} = \frac{D(e(1/6))i}{8} = -V = -0.1268677\cdots i. \tag{11.32}$$

With the definition of $\psi_{-1}(4z;-e^{-2iv})$ from Lemma 4.4.1, the power series $\gamma^-_{(1/4)}(z)\in\mathbb{C}[[z]]$ is defined as a formal Gaussian integration by

$$\begin{aligned}\gamma^-_{(1/4)}(z) &= \frac{1}{2\sqrt{-2\pi i}}e^{-\pi i/24-z/12}\\ &\times\int\exp\Bigg(-\sqrt{3}i\frac{s^2}{2}+\frac{is\sqrt{z}}{2}+\frac{1}{8}\sum_{l\geq 3}\mathrm{Li}_{2-l}(Q)\frac{(4s\sqrt{z})^l}{zl!}\\ &-\sum_{t=1}^{2}\sum_{k\geq 1,r\geq 0}B_k\left(1-\frac{t}{2}\right)\mathrm{Li}_{2-k-r}(-(-1^t)\sqrt{Q})\frac{(2s\sqrt{z})^r(4z)^{k-1}}{r!k!}\Bigg)ds.\end{aligned} \tag{11.33}$$

The first coefficients are given by

$$\gamma^-_{(1/4)}(z) = \frac{1}{2\sqrt[4]{3(2-\sqrt{3})}}\left(1+\left(\frac{1}{3}+\frac{77}{216}\sqrt{3}\right)iz-\left(\frac{89449}{31104}+\frac{647}{648}\sqrt{3}\right)z^2+O(z^3)\right). \tag{11.34}$$

Putting everything together, we obtain the asymptotic expansion

$$v_1^{[0]}(q) = \sqrt{\frac{2\pi i}{-z}}e^{-V/z}\,\gamma^-_{(1/4)}(z)(1+O(z^L)) + \phi^{[0]}_{(1/4)}(z)(1+O(z^L)) \tag{11.35}$$

for all $L>0$. If $\arg\varphi>0$, the exponential contribution is the biggest term in (11.6) and for $\arg\varphi<0$, the power series $\phi^{[0]}_{(1/4)}(z)$ has the largest contribution. □

Sum over odd parts

Proof of (11.7). The asymptotics of $v_1^{[1]}(q)$ as defined in (11.46) as $q\to i$ is similar.

We change the contour in the integral representation in Lemma 11.1.3 to a stationary contour $\mathcal{S}$. After applying the asymptotics from Lemma 4.4.2 and Lemma 4.4.1 and following a similar argument as in subsection 11.1.2, we obtain that $v_1^{[1]}(q)$ is asymptotically equal to

$$\begin{aligned}&\frac{-e^{\pi i3/4+\pi^2/48z-z/12}}{2z}\int_{\mathcal{S}}\mathrm{sign}(\mathrm{Re}(v/\varphi))\exp\left(\frac{-\mathrm{Li}_2(e^{-4iv})+4v^2+2\pi v-4\,\mathrm{sign}(\mathrm{Re}(v/\varphi))\pi v}{8z}\right)\\ &\qquad\exp\left(-\frac{iv}{2}+\frac{\mathrm{Li}_1^{\varphi}(e^{-2iv})}{2}+\psi_{-1}(4z;e^{-2iv})\right)dv+\phi^{[1]}_{(1/4)}(z),\end{aligned} \tag{11.36}$$

where

$$\begin{aligned}\phi^{[1]}_{(1/4)}(z) &= \sum_{\substack{n<0\\ n\equiv 1 \bmod 2}}\frac{q^{n(n+1)/2}}{(-q^2;q^2)_n} = \sum_{\substack{l>0:\\ l\equiv 1 \bmod 2}}q^{-l(l-1)/2}(-1;q^2)_l\\ &= 2\sum_{\substack{m\geq 0:\\ m\equiv 0 \bmod 2}}q^{-m(m+1)/2}(-q^2;q^2)_m\\ &= 2+8iz-96\,z^2-\frac{5708}{3}i\,z^3+52640\,z^4+\frac{28056121}{15}i\,z^5-\frac{405909568}{5}z^6\\ &\quad-\frac{2622584263067}{630}i\,z^7+\frac{5171242573856}{21}z^8+\frac{7487418817497741041}{45360}i\,z^9+O(z^{10}).\end{aligned} \tag{11.37}$$

We define

$$\begin{aligned} f(v) &= -\frac{\mathrm{Li}_2^{\varphi}(e^{-4iv})}{8} + \frac{v^2}{2} + \frac{\pi v}{4} - \operatorname{sign}\left(\mathrm{Re}\left(v/\varphi\right)\right)\frac{\pi v}{2}, \\ g(z;v) &= \operatorname{sign}(\mathrm{Re}(v/\varphi))\exp\left(-\frac{iv}{2} + \frac{\mathrm{Li}_1^{\varphi}(e^{-2iv})}{2} + \psi_{-1}(4z;e^{-2iv})\right), \end{aligned} \tag{11.38}$$

where f and g are holomorphic functions on the domain defined in (11.29) for the same reason as in subsection 11.1.2. Then

$$v_1^{[1]}(q) = \frac{-e^{\pi i 3/4}}{2z} e^{\pi^2/48z - z/12} \int_{\mathcal{S}} e^{f(v)/z} g(z;v) dv (1 + O(|z|^L)) + \phi^{[0]}_{(1/4)}(z)(1 + O(z^L)) \tag{11.39}$$

for all $L > 0$ and we compute

$$\begin{aligned} f'(v) &= -\frac{i}{2}\log(1 - e^{-4iv}) + v + \frac{\pi v}{4} - \operatorname{sign}\left(\mathrm{Re}\left(v/\varphi\right)\right)\frac{\pi v}{2} \\ f''(v) &= \frac{1 + e^{-4iv}}{1 - e^{-4iv}} \end{aligned} \tag{11.40}$$

such that the unique stationary point of f is $v_0 = \frac{\pi}{12}$. Similar arguments as above using the saddle-point method imply that the contribution corresponding to the stationary point $v_0 = \frac{\pi}{12}$ is given by

$$\begin{aligned} &e^{\pi^2/48z + f(v_0)/z - z/12} \int \exp\Big(-f''(\tfrac{\pi}{12})\frac{s^2}{2} + \sum_{l\geq 3} \frac{f^{(l)}(Q)}{l!}(is)^l z^{l/2-1}\Big) g\Big(\frac{\pi}{12} + is\sqrt{z}; z\Big) ds \\ &= e^{V/z}\sqrt{\frac{2\pi i}{z}}\, \gamma^{+}_{(1/4)}(z) \end{aligned} \tag{11.41}$$

where

$$\begin{aligned} \gamma^{+}_{(1/4)}(z) &= \frac{1}{2\sqrt{2\pi}} e^{-\pi i/24 - z/12} \\ &\quad\times \int \exp\Big(\sqrt{3}i\frac{s^2}{2} + \frac{is\sqrt{z}}{2} + \frac{1}{8}\sum_{l\geq 3} \mathrm{Li}_{2-l}(Q)\frac{(4s\sqrt{z})^l}{zl!} \\ &\qquad - \sum_{t=1}^{2} \sum_{k\geq 1, r\geq 0} B_k\Big(1 - \frac{t}{2}\Big) \mathrm{Li}_{2-k-r}((-1^t)\sqrt{Q}) \frac{(2s\sqrt{z})^r (4z)^{k-1}}{r!k!}\Big) ds. \end{aligned} \tag{11.42}$$

The first coefficients are given by

$$\gamma^{+}_{(1/4)}(z) = \frac{1}{2\sqrt[4]{3(2-\sqrt{3})}} \left(1 + \left(\frac{1}{3} - \frac{77}{216}\sqrt{3}\right) iz + \left(-\frac{89449}{31104} + \frac{647}{648}\sqrt{3}\right) z^2 + O(z^3)\right). \tag{11.43}$$

Putting everything together, we obtain the asymptotic expansion

$$v_1^{[1]}(q) = \sqrt{\frac{2\pi i}{z}} e^{V/z}\, \gamma^{+}_{(1/4)}(z)(1 + O(|z|^L)) + \phi^{[1]}_{(1/4)}(z)(1 + O(z^L)) \tag{11.44}$$

for all $L > 0$ which completes the proof. $\square$

11.1.3. Proof of Theorem 10.2.2, (2)

Throughout, let $\zeta = e(\alpha) = e(r/m)$ be a root of unity of order m divisible by 4. As in subsection 11.1.2, we split up the sum defining $v_1(q)$ depending on $n \bmod m$

$$v_1(q) = \sum_{n_0=0}^{\frac{m}{2}-1} v_1^{[n_0]}(q), \tag{11.45}$$

where

$$v_1^{[n_0]}(q) = \sum_{\substack{n\geq 0 \\ n\equiv n_0 \bmod \frac{m}{2}}} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n}. \tag{11.46}$$

for $n_0 \in \{0, \dots, \frac{m}{2}-1\}$. We start with the following lemma.

Lemma 11.1.4. *Let $\zeta = e(\alpha) = e\left(\frac{r}{m}\right)$ be as above.*

1. *Choose $\overline{r} \in \mathbb{Z}$ with $\overline{r} = r \bmod 4$. If $n = n_0 \bmod \frac{m}{2}$ then*

$$\zeta^{n(n+1)/2} = \zeta^{n_0(n_0+1)/2} e\left(\frac{n-n_0}{4} + (-1)^{n_0}\frac{\overline{r}}{2}\frac{n-n_0}{m}\right). \tag{11.47}$$

2. *Let $\overline{\overline{r}} = \pm 1$ such that $\overline{\overline{r}} = r \bmod 4$. Then for all $n = n_0 \bmod \frac{m}{2}$*

$$\zeta^{n(n+1)/2}(-1)^{2(n-n_0)/m} = \zeta^{n_0(n_0+1)/2} e\Big((-1)^{m/4+n_0+1}\ \overline{\overline{r}}\ \frac{n-n_0}{2m}\Big). \tag{11.48}$$

Proof. We begin by proving part (1).

1. We write with $n = km/2 + n_0$, $k = 2\frac{n-n_0}{m}$,

$$\begin{aligned}
\zeta^{n(n+1)/2} &= e\left(\alpha n(n+1)/2\right) \\
&= e\left(\alpha\left(\tfrac{km}{2}+n_0\right)\left(\tfrac{km}{2}+n_0+1\right)/2\right) \\
&= e\left(\alpha\left(\tfrac{km}{2}+n_0\right)^2/2 + \alpha\left(\tfrac{km}{2}+n_0\right)/2\right) \\
&= e\left(\alpha\left(\tfrac{k^2m^2}{8}+\tfrac{kmn_0}{2}+\tfrac{n_0^2}{2}+\tfrac{km}{4}+\tfrac{n_0}{2}\right)\right).
\end{aligned} \tag{11.49}$$

Note that $\frac{\alpha m^2}{8} \in \frac{1}{2}\mathbb{Z}$ and thus $e(\alpha\frac{k^2m^2}{8}) = e(\alpha\frac{km^2}{8})$ as $k^2 = k \bmod 2$. Hence, we obtain

$$\begin{aligned}
\zeta^{n(n+1)/2} &= e\left(\alpha k\left(\frac{m^2}{8}+\frac{mn_0}{2}+\frac{m}{4}\right)\right)\ e\left(\alpha\frac{n_0(n_0+1)}{2}\right) \\
&= e\left(\alpha km\left(\frac{m}{8}+\frac{n_0}{2}+\frac{1}{4}\right)\right)\ \zeta^{n_0(n_0+1)/2}.
\end{aligned} \tag{11.50}$$

Moreover, we compute

$$e\left(\alpha km\left(\frac{m}{8}+\frac{n_0}{2}+\frac{1}{4}\right)\right) = e\left(\alpha\frac{km}{4}\left(\frac{m}{2}+2n_0+1\right)\right) \tag{11.51}$$

and note that the denominator of $\alpha\frac{km}{4}$ is either 1 or 4. If n_0 is even, $2n_0$ is divisible by 4 and $e(\alpha\frac{km}{4}2n_0)) = 1$. Otherwise, $2n_0 + 2$ is divisible by 4 and thus $e(\alpha\frac{km}{4}(2n_0+1)) = e(-\alpha\frac{km}{4})$. In other words,

$$\begin{aligned} e\left(\alpha\frac{km}{4}\left(\frac{m}{2}+2n_0+1\right)\right) &= \begin{cases} e\left(\alpha\frac{km}{4}\left(\frac{m}{2}+1\right)\right), & \text{if } n_0 \text{ is even,} \\ e\left(\alpha\frac{km}{4}\left(\frac{m}{2}-1\right)\right), & \text{if } n_0 \text{ is odd,} \end{cases} \\ &= e\left(\alpha\frac{km}{4}\left(\frac{m}{2}+(-1)^{n_0}1\right)\right) \\ &= e\left(\alpha\frac{km^2}{8}\right)\, e\left((-1)^{n_0}\alpha\frac{mk}{4}\right). \end{aligned} \tag{11.52}$$

Note that $\alpha\frac{km^2}{8} \in \frac{1}{2}\mathbb{Z}$, hence

$$e\left(\alpha\frac{km^2}{8}\right) \;=\; e\left(\frac{km}{8}\right) \;=\; e\left(\frac{n-n_0}{4}\right). \tag{11.53}$$

Moreover, we note that $\alpha\frac{mk}{4}$ has denominator 4 such that

$$e\Big((-1)^{n_0}\alpha\frac{mk}{4}\Big) = e\Big((-1)^{n_0}\overline{r}\frac{k}{4}\Big) = e\Big((-1)^{n_0}\frac{\overline{r}}{2}\frac{n-n_0}{m}\Big). \tag{11.54}$$

This proves the first part of the lemma.

2. To prove part (2), we choose $\overline{r}$ with $\overline{r} = \overline{\overline{r}} + 4l$, $l \in \mathbb{Z}$ such that

$$(-1)^{n_0}2l + \frac{m}{4} - 1 \;=\; \begin{cases} 0, & \text{if } \frac{m}{4} \text{ is odd,} \\ -(-1)^{n_0}\overline{\overline{r}}, & \text{if } \frac{m}{4} \text{ is even.} \end{cases} \tag{11.55}$$

We continue with the notation from above with $(-1)^{2(n-n_0)/m} = e(-\frac{k}{2})$:

$$\begin{aligned} e\left(\tfrac{km}{8}\right) e\left((-1)^{n_0}\tfrac{\overline{r}k}{4}\right) e\left(-\tfrac{k}{2}\right) &= e\left(\tfrac{km}{8} + (-1)^{n_0}\tfrac{\overline{r}k}{4} - \tfrac{k}{2}\right) \\ &= e\left(k\ \left(\tfrac{m}{4} + (-1)^{n_0}\tfrac{\overline{r}}{2} - 1\right)/2\right) \\ &= e\left(k\ \left(\tfrac{m}{4} + (-1)^{n_0}\tfrac{\overline{\overline{r}}+4l}{2} - 1\right)/2\right) \\ &= e\left(k\ \left(\tfrac{m}{4} + (-1)^{n_0}\tfrac{\overline{\overline{r}}}{2} + (-1)^{n_0}2l - 1\right)/2\right). \end{aligned} \tag{11.56}$$

Using the choice of $\overline{r}$, the last expression becomes

$$\begin{cases} e\left(k\ (-1)^{n_0}\frac{\overline{\overline{r}}}{4}\right), & \text{if } \frac{m}{4} \text{ is odd,} \\ e\left(k\ \left((-1)^{n_0}\frac{\overline{\overline{r}}}{2} - (-1)^{n_0}\overline{\overline{r}}\right)/2\right), & \text{if } \frac{m}{4} \text{ is even,} \end{cases} \tag{11.57}$$

which in both cases is equal to

$$e\left(k\ (-1)^{m/4-1+n_0}\ \frac{\overline{\overline{r}}}{4}\right) \;= e\left((-1)^{m/4-1+n_0}\ \overline{\overline{r}}\frac{n-n_0}{2m}\right). \tag{11.58}$$

□

From now on we write

$$\delta \;:=\; (-1)^{m/4+n_0+1}\ \overline{\overline{r}} \tag{11.59}$$

with $\overline{\overline{r}} = \pm 1$ as in the previous lemma.

Integral representation

Lemma 11.1.5. *Let L_∞ be the contour depicted in Figure 11.1. Then with $q = \zeta e^{-z}$,*

$$v_1^{[n_0]}(q) \;=\; \frac{-\zeta^{n_0(n_0+1)/2} e(\delta \frac{n_0}{2m})}{m(-q^2;q^2)_\infty} \int_{L_\infty} e^{\pi\delta t/m} e^{zt^2/2} e^{-izt/2} \frac{(-\zeta^{2n_0} e^{-2zit} q^2; q^2)_\infty}{\sin(\pi 2(s-n_0)/m)} dt. \tag{11.60}$$

Proof. First, note that we have

$$v_1^{[n_0]}(q) \;=\; \frac{\zeta^{n_0(n_0+1)/2}}{(-q^2;q^2)} \sum_{\substack{n\geq 0\\ n=n_0 \bmod \frac{m}{2}}} e(\delta \tfrac{n-n_0}{2m}) e^{-hn(n+1)/2} \, (-\zeta^{2n_0} e^{-h2n} q^2; q^2)_\infty \tag{11.61}$$

using Lemma 11.1.4, (2). The function $\frac{1}{\sin(2\pi(s-n_0)/m)}$ has poles at $s \in \mathbb{Z}$ with $s = n_0 \bmod \frac{m}{2}$ and residues $(-1)^{2(n-n_0)/m} \, \frac{m}{2\pi}$. Hence, we write with Cauchy's residue theorem

$$\begin{aligned} v_1^{[n_0]}(q) \;&=\; \frac{-1}{2\pi i} \frac{1}{(-q^2;q^2)_\infty} \\ &\quad \lim_{R\to\infty} \int_{L_R + C_R} \zeta^{s(s+1)/2} e^{-hs(s+1)/2} (-\zeta^{2n_0} e^{-2hs} q^2; q^2)_\infty \frac{2\pi(-1)^{2(s-n_0)/m}}{m \sin(\pi 2(s-n_0)/m)} ds \\ &= \frac{-\zeta^{n_0(n_0+1)/2} e(\delta \frac{n_0}{2m})}{im(-q^2;q^2)_\infty} \lim_{R\to\infty} \int_{L_R + C_R} e^{\pi i \delta s/m} e^{-hs(s+1)/2} \frac{(-\zeta^{2n_0} e^{-2hs} q^2; q^2)_\infty}{\sin(\pi 2(s-n_0)/m)} ds. \end{aligned}$$

The convergence follows analogously to Lemma 11.1.3. $\square$

Proof of Theorem 10.2.2, (2)

The proof of Theorem 10.2.2, (2) and the asymptotics of $v_1(q)$ follow from the following proposition by summing $v_1^{[n_0]}(q)$ over $n_0 = 0, \ldots, \frac{m}{2} - 1$ as in (11.45) and setting

$$\gamma^{\pm}_{(\alpha)} \;=\; \sum_{\substack{n_0 = 0,\ldots,\frac{m}{2}\\ \delta = \pm 1}} \gamma^{[n_0]}_{(\alpha)}. \tag{11.62}$$

Proposition 11.1.6. *Let $\zeta = e(\alpha)$ be a root of unity of order m. For $n_0 \in \{0, \ldots, \frac{m}{2} - 1\}$, we have*

$$v_1^{[n_0]}(q) \;=\; e^{\delta \frac{16V}{zm^2}} \left(\frac{z}{2\pi i}\right)^{-1/2} \gamma^{[n_0]}_{(\alpha)}(z)(1 + O(|z|^L)) + \phi^{[n_0]}_{(\alpha)}(z)(1 + O(|z|^L)) \tag{11.63}$$

for all $L > 0$ as $q = \zeta e^{-z} \to \zeta$, where $\gamma^{[n_0]}_{(\alpha)}(z) \in \mathbb{C}[[z]]$ is defined in (11.72) and

$$\phi^{[n_0]}_{(\alpha)}(z) \;=\; \sum_{\substack{n<0\\ n\equiv n_0 \bmod \frac{m}{2}}} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n}. \tag{11.64}$$

Proof. Throughout, we write $z = \varphi h$ where $\varphi \in \mathbb{C}$ with $|\varphi| = 1$ and $0 \neq |\arg(\varphi)| < \frac{\pi}{2}$. We substitute $s = iv/z$ in the integral representation from Lemma 11.1.5 to obtain

$$v_1^{[n_0]}(q) \;=\; \frac{-\zeta^{n_0(n_0+1)/2} e(\delta \frac{n_0}{2m})}{mh(-q^2;q^2)_\infty} \int_{-ihL_\infty} e^{\pi\delta v/mz} e^{v^2/2z} e^{-iv/2} \frac{(-\zeta^{2n_0} e^{-2iv} q^2; q^2)_\infty}{\sin(\pi 2(iv/z - n_0)/m)} dv. \tag{11.65}$$

Changing the contour of integration to a stationary contour $\mathcal{S}$, we include the poles at $-2iz\mathbb{Z}_{<0}$ whose residues give a power series

$$\phi^{[n_0]}_{(\alpha)}(z) \;=\; \sum_{\substack{n<0\\ n\equiv n_0 \bmod \frac{m}{2}}} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n}. \tag{11.66}$$

Applying the asymptotics from Lemma 4.4.2 and Lemma 4.4.1 to the integrands, we obtain that $v_1^{[n_0]}(q)$ is equal to

$$\frac{-\zeta^{n_0(n_0+1)/2}e(\delta\frac{n_0}{2m})}{mhQ(\zeta^2)}\int_{\mathcal{S}} e^{f(v)/z}g(z;v)dv + \phi^{[n_0]}_{(\alpha)}(z) \tag{11.67}$$

to all orders where $Q(\zeta^2)$ is defined in (4.44) and

$$\begin{aligned} f(v) &= -\frac{2\,\mathrm{Li}_2^{\varphi}(e^{-miv})}{m^2}+\frac{v^2}{2}-\frac{\pi\delta v-\operatorname{sign}(\operatorname{Re}(v/\varphi))\,2\pi v}{m},\\ g(z;v) &= \operatorname{sign}(\operatorname{Re}(v/\varphi))\exp\Bigg(-\frac{iv}{2}-\operatorname{sign}(\operatorname{Re}(v/\varphi))2\pi i m n_0-\frac{\mathrm{Li}_1^{\varphi}(e^{-miv})}{2}\\ &\qquad +\sum_{t=1}^{m/2}\frac{2t}{m}\,\mathrm{Li}_1^{\varphi}(-\zeta^{2t+2n_0}e^{-2iv})+\psi_{\zeta^2}(mz;-\zeta^{2n_0}e^{-2iv})\Bigg). \end{aligned} \tag{11.68}$$

We recall that $\mathrm{Li}_2^{\varphi}(e^{-miv})$ jumps by $2\pi mv$ when v crosses the branch cut at $\operatorname{Re}(v/\varphi)=0$. Hence, the function $f(v)$ is holomorphic on the domain defined in (11.29). A similar argument shows that $g(v)$ is holomorphic on the same domain.

The stationary points v_0 of f are given by

$$f'(v_0) \;=\; -\frac{2i\log(1-e^{-miv_0})}{m}+v_0-\frac{\pi\delta-\operatorname{sign}(\operatorname{Re}(v_0/\varphi))\,2\pi}{m} \;=\; 0. \tag{11.69}$$

This implies in particular $(1-e^{-iv_0m})^2=-e^{-iv_0m}$, i.e., $e^{-miv_0}=e(\pm 1/6)$ and it can be checked that $v_0=\delta\frac{\pi}{3m}$ is the unique stationary point.

Applying the saddle-point method and using $\frac{\pi^2}{3m^2}+f(v_0) \;=\; \delta\frac{16V}{m^2}$ implies that the contribution coming from the saddle point is given by

$$\begin{aligned} &\frac{-\zeta^{n_0(n_0+1)/2}e(\delta\frac{n_0}{2m})}{mhQ(\zeta^2)\sqrt{z}}e^{\delta 16V/m^2z-z/12}\int\exp\Bigg(-f''(v_0)\frac{s^2}{2}+\sum_{l\geq 3}\frac{f^{(l)}(e^{-miv_0})}{l!}(is)^l z^{l/2-1}\Bigg)\\ &\qquad\qquad g\Big(v_0+is\sqrt{z};z\Big)ds\\ &= e^{\frac{\delta V}{zm^2}}\sqrt{\frac{2\pi}{\delta z}}\gamma^{[n_0]}_{(\alpha)}(z) \end{aligned} \tag{11.70}$$

where the integral goes through a small neighbourhood of v_0. In other words, we have the asymptotic expansion

$$v_1^{[n_0]}(q) \;=\; e^{\frac{\delta V}{zm^2}}\left(\frac{\delta z}{2\pi}\right)^{-1/2}\gamma^{[n_0]}_{(\alpha)}(z)(1+O(|z|^L))+\phi^{[n_0]}_{(\alpha)}(z)(1+O(|z|^L)) \tag{11.71}$$

for all $L>0$ where

$$\gamma^{[n_0]}_{(\alpha)} \;=\; -\frac{\zeta^{n_0(n_0+1)/2}e(\delta\frac{n_0}{2m})2g(v_0)}{mQ(\zeta^2)\sqrt{f''(v_0)}} \tag{11.72}$$

with
$$f''(v_0) \;=\; \frac{1+e^{-miv_0}}{1-e^{-miv_0}}. \tag{11.73}$$
This completes the proof. □

11.2. Proof of Theorem 10.2.3: The asymptotics of $V_1(n)$

We summarise the proof of Theorem 10.2.3, following [FMRS23].

Proof of Theorem 10.2.3. Following the circle method, we use Cauchy's integral formula to write
$$V_1(n) \;=\; \frac{1}{2\pi i}\int_C \frac{v_1(q)}{q^{n+1}}dq, \tag{11.74}$$
where C denotes a circle of radius < 1 going counterclockwise around 0. We split up C into C_+, C_-, and C', where $C_\pm = \{\pm ie^{-\lambda+i\theta}, \theta \in (-\delta,\delta)\}$ for $\lambda = \sqrt{\frac{|V|}{n}}$ and some $\delta > 0$ and $C' = C \setminus C_\pm$.

We parameterise C_+ as $q = ie^{-z}$ where z is between $\lambda + i\delta$ and $\lambda - i\delta$. Then
$$\frac{1}{2\pi i}\int_{C_+} \frac{v_1(q)}{q^{n+1}}dq \;=\; \frac{(-i)^n}{2\pi i}\int_{\lambda-i\delta}^{\lambda+i\delta} v_1(ie^{-z})e^{zn}dz. \tag{11.75}$$
Recall from Theorem 10.2.2 that we have
$$v_1(ie^z) \;=\; e^{V/z}\left(\frac{z}{2\pi i}\right)^{-1/2}\gamma^+(1+O(z)) + e^{-V/z}\left(\frac{-z}{2\pi i}\right)^{-1/2}\gamma^-(1+O(z)) \tag{11.76}$$
as $z \to 0$ in the right half-plane. Therefore, the integral in (11.75) with $\delta = \lambda = \sqrt{\frac{|V|}{n}}$ is asymptotically equal to
$$\begin{aligned}&\frac{(-i)^n}{\sqrt{2\pi i}}\int_{\lambda-i\delta}^{\lambda+i\delta}\left(\gamma^+e^{V/z} - i\gamma^-e^{-V/z}\right)z^{-1/2}e^{zn}dz \qquad\qquad z \mapsto \frac{z}{\sqrt{n}}\\ &= \frac{(-i)^n}{n^{1/4}\sqrt{2\pi i}}\int_{\sqrt{|V|}(1-i)}^{\sqrt{|V|}(1-i)}\left(\gamma^+e^{\sqrt{n}(V/z+z)} - i\gamma^-e^{\sqrt{n}(-V/z+z)}\right)z^{-1/2}dz\end{aligned} \tag{11.77}$$
with an error of order $O(n^{-1})$. If we change the contour of integration through a saddle point $\pm\sqrt{V}$, i.e., a zero of the derivative of $\pm\frac{V}{z} + z$, we can make the change of variables $z \mapsto \pm\sqrt{V} + izn^{-1/4}$. If we expand the integral near $z = 0$, we obtain that the first term in (11.76) is asymptotically equal to
$$\gamma^+\frac{i(-i)^n}{\sqrt{2in}}e^{2\sqrt{Vn}}\,(1+O(n^{-\frac{1}{2}})). \tag{11.78}$$
Similar computations show that the second term in (11.76) is asymptotically equal to
$$\gamma^-\frac{i(-i)^n}{\sqrt{2in}}e^{2\sqrt{-Vn}}\,(1+O(n^{-\frac{1}{2}})). \tag{11.79}$$
Summing up, we note that the integral in (11.75) is asymptotically given by
$$\begin{aligned}V_1(n) \;=\; \frac{1}{2\pi i}\int_{C_+}\frac{v_1(q)}{q^{n+1}}dq \;&= \gamma^+\frac{\sqrt{i}(-i)^n}{\sqrt{2n}}e^{2\sqrt{Vn}}\,(1+O(n^{-\frac{1}{2}}))\\ &+\gamma^-\frac{\sqrt{i}(-i)^n i}{\sqrt{2n}}e^{2\sqrt{-Vn}}\,(1+O(n^{-\frac{1}{2}})).\end{aligned} \tag{11.80}$$

Similarly, the contribution coming from C_- is given by

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_+}\frac{v_1(q)}{q^{n+1}}dq &= \gamma^+\frac{\sqrt{i}\,i^{n-1}}{\sqrt{2n}}e^{2\sqrt{-Vn}}\left(1+O(n^{-\frac{1}{2}})\right) \\ &\quad -\gamma^-\frac{\sqrt{i}\,i^{n}}{\sqrt{2n}}e^{2\sqrt{Vn}}\left(1+O(n^{-\frac{1}{2}})\right).\end{aligned}\tag{11.81}$$

Combining (11.80) and (11.81), we see that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{C_+\cup C_-}\frac{v_1(q)}{q^{n+1}}dq \\ &= \left(\gamma^+\frac{\sqrt{i}(-i)^n}{\sqrt{2n}}e^{2\sqrt{Vn}}+\gamma^-\frac{\sqrt{i}(-i)^n i}{\sqrt{2n}}e^{2\sqrt{-Vn}}+\gamma^+\frac{\sqrt{i}\,i^{n-1}}{\sqrt{2n}}e^{2\sqrt{-Vn}}-\gamma^-\frac{\sqrt{i}\,i^{n}}{\sqrt{2n}}e^{2\sqrt{Vn}}\right) \\ &\quad\times(1+O(n^{\frac{1}{2}}).\end{aligned}\tag{11.82}$$

Using $\sqrt{\pm V}=(1\pm i)\sqrt{|V|}$, we can rewrite the expression as

$$\begin{aligned}\frac{\sqrt{i}\,i^n}{\sqrt{2n}}e^{\sqrt{2|V|n}}\Big(&\gamma^+\left(((-1)^n-i)\cos(\sqrt{2|V|n})+(i(-1)^n-1)\sin(\sqrt{2|V|n}))\right) \\ &+\gamma^-\left((i(-1)^n-1)\cos(\sqrt{2|V|n})+((-1)^n-i)\sin(\sqrt{2|V|n})\right)\Big).\end{aligned}\tag{11.83}$$

Note that $\sqrt{i}\,i^n((-1)^n-i)=\sqrt{2}(-1)^{\lfloor\frac{n}{2}\rfloor}$ and $\sqrt{i}\,i^n(i(-1)^n-1)=-\sqrt{2}(-1)^{\lfloor\frac{n}{2}\rfloor}(-1)^n$ such that we can rearrange the terms to obtain that

$$(-1)^{\lfloor\frac{n}{2}\rfloor}\frac{e^{\sqrt{2|V|n}}}{\sqrt{n}}(\gamma^+-(-1)^n\gamma^-)(\cos(\sqrt{2|V|n})-(-1)^n\sin(\sqrt{2|V|n})),\tag{11.84}$$

which agrees with the leading term in the asymptotics in (10.22).

It remains to estimate the integral over C'. According to Theorem 10.2.2, the largest contributions of the integral over C' come from the roots of unity of order 8. Hence, we can bound the complete integral over C' by the contribution coming from the roots of unity of order 8 times the length of the integral. Applying the same ideas as for the major arcs, we obtain that the contribution in the integral near a root of unity ζ of order $m>4$ with $4|m$ is given by

$$\frac{1}{\sqrt{2\pi i}\zeta^n}\int_{\lambda-i\delta}^{\lambda+i\delta}e^{\pm\frac{16V}{m^2z}+nz}z^{-\frac{1}{2}}dz.\tag{11.85}$$

This leads to a contribution of

$$e^{2\sqrt{\pm\frac{16nV}{m^2}}}n^{-\frac{1}{2}}\left(1+O\left(n^{-\frac{1}{2}}\right)\right),\tag{11.86}$$

up to a multiplicative factor. We can estimate the integral over C' by the contribution coming from the roots of unity of order $m=8$ and thus

$$\frac{1}{2\pi i}\int_{C'}\frac{v_1(q)}{q^{n+1}}dq \ll |e^{\sqrt{\pm nV}}n^{-\frac{1}{2}}| = O\left(e^{\sqrt{\frac{n|V|}{2}}}n^{-\frac{1}{2}}\right).\tag{11.87}$$

This gives the error term in (10.22) which completes the proof.

□

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