PERTURBATIVE INVARIANTS OF CUSPED HYPERBOLIC 3-MANIFOLDS

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Abstract. We prove that a formal power series associated to an ideally triangulated cusped hyperbolic 3-manifold (together with some further choices) is a topological invariant. This formal power series is conjectured to agree to all orders in perturbation theory with two important topological invariants of hyperbolic knots, namely the Kashaev invariant and the Andersen–Kashaev invariant (also known as the state-integral) of Teichmüller TQFT.

Contents

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1. Introduction

This paper concerns the topological invariance of a formal power series associated to an ideally triangulated 3-manifold M with a torus boundary component [11]. The series is defined by formal Gaussian integration of a finite dimensional integral (a so-called "state-integral") and it is expected to coincide to all orders with the asymptotic expansion of three important topological invariants of 3-manifolds.

The first is the Kashaev invariant of a hyperbolic knot [25], where Kashaev's famous volume conjecture is refined to an asymptotic statement to all orders in perturbation theory using the above formal power series. This was studied in detail in [21] based on extensive numerical computations, but it is only proven for a handful of hyperbolic knots.

The second invariant is the Andersen–Kashaev state integral [2], whose asymptotic expansion for the simplest hyperbolic 4_1 knot was shown to agree with the above series in [2, Sec.12], and also observed numerically for the first three simplest hyperbolic knots in [20]. The state-integrals of [2] are finite-dimensional integrals whose integrand is a product of Faddeev quantum dilogarithms times an exponential of a quadratic form, assembled from an ideal triangulation of a 3-manifold with torus boundary components. Andersen–Kashaev proved that these are topological invariants that are the partition function of the Teichmüller TQFT [2, 1], which is a 3-dimensional version of a quantization of Teichmüller space [26, 17].

A third invariant is the Chern–Simons theory with complex gauge group $\mathrm{SL}_2(\mathbb{C})$. This theory was introduced by Witten [38] and studied extensively by Gukov [23]. Although Chern–Simons theory with compact gauge group $\mathrm{SU}(2)$ has an exact nonperturbative definition given by the so-called Witten–Reshetikhin–Turaev invariant [37, 33] and a well–defined perturbation theory involving Feynman diagrams with uni-trivalent vertices [3, 4], the same is not known for Chern–Simons theory with complex gauge group $\mathrm{SL}_2(\mathbb{C})$. For reasons that are not entirely understood, the partition function of complex Chern–Simons theory for 3-manifolds with torus boundary reduces to a finite-dimensional state-integral, as if some unknown localization principle holds. The corresponding state-integrals were introduced and studied by Hikami [24], Dimofte [10] and others [12]. Unfortunately, in those works the integration contour was not pinned down, and this problem was finally dealt with in [2] and, among other things, implied topological invariance of the state-integrals, which were coined to be the partition function of Teichmüller TQFT. But ignoring the contour of integration, and focusing on a critical point of the action, which is a solution to a system of polynomial equations, allowed [11] to give a definition of the formal power series that are the main focus of our paper. Note, however, that the Feynman diagrams of [11] involve stable graphs of arbitrary valency, a perturbative definition of Chern–Simons theory with complex gauge groups would involve uni-trivalent graphs.

The above discussion points out several aspects of these formal power series and conjectural relations to perturbation theory of complex Chern–Simons and of Teichüller TQFT. Aside from conjectures, this paper concerns a theorem, the topological invariance of the above formal power series.

Let us briefly recall the key ingredients that go into the definition of the series, and discuss those in detail in later sections. The first ingredient is an ideal triangulation $\mathcal{T}$ with N ideal tetrahedra of a 3-manifold M with torus boundary. Each tetrahedron has shapes $z \in \mathbb{C} \setminus \{0,1\}$, $z' = 1/(1-z)$ and $z'' = 1 - 1/z$ attached to its three pairs of opposite edges, and the shapes satisfy a system of polynomial equations (so-called "gluing equations" [34] or Neumann-Zagier equations [29]) determined by the combinatorics of the triangulation, one for each edge and peripheral curve. After some choices are made (such as an ordering of the tetrahedra and their edges, a choice of shape for each tetrahedron, a choice of an edge to remove from the gluing equations, and a choice of peripheral curve to include), one obtains two matrices A and B with integer entries such that $(A\,|\,B)$ is the upper-half of a symplectic $2N \times 2N$ matrix, as well as a vector $\nu \in \mathbb{Z}^N$. In addition, we choose a solution $z = (z_1, \dots, z_N)$ of the gluing equations as well as a flattening (f, f''), i.e., an integer solution to the equation $\nu = Af + Bf''$. The power series $\Phi^{\Xi}(\hbar)$ depends on the tuple $\Xi = (A, B, \nu, z, f, f'')$, which we collectively call a NZ-datum.

The next ingredient that goes in the definition of $\Phi^{\Xi}(\hbar)$ is an auxiliary formal power series

$$\psi_\hbar(x,z) \;:=\; \exp\Bigg(-\sum_{\substack{k,\ell\in\mathbb{Z}_{\geq 0}\\ k+\frac{\ell}{2}>1}} \frac{B_k\, x^\ell\, \hbar^{k+\frac{\ell}{2}-1}}{\ell!\,k!}\,\mathrm{Li}_{2-k-\ell}(z)\Bigg) \in 1 + \hbar^{\frac12}\mathbb{Q}[z,(1-z)^{-1}][x][\![\hbar^{\frac12}]\!]\,. \tag{1}$$

This series, which differs from the one given in [11, Eq. 1.9] by a factor of $\exp\big(\frac{1}{12}\hbar - \frac12 x\hbar^{\frac12}\big)$, comes from the asymptotics of the infinite Pochhammer symbol (also known as the quantum dilogarithm)

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x) = \exp\Big(-\sum_{n=1}^{\infty}\frac{x^n}{n(1-q^n)}\Big)\,. \tag{2}$$

Explicitly, for complex numbers x and z with $|z|<1$ and $\mathrm{Re}(\hbar)<0$ we have (see [39] and also [22, Lem.2.1])

$$(ze^{x\hbar^{1/2}};e^\hbar)_\infty^{-1} \;\sim\; \frac{1}{\sqrt{1-z}}\exp\Big(-\frac{\mathrm{Li}_2(z)}{\hbar} - \frac{\mathrm{Li}_1(z)x}{\hbar^{\frac12}} - \frac{\mathrm{Li}_0(z)x^2}{2}\Big)\psi_\hbar(x,z)\,, \qquad (\hbar\to 0)\,. \tag{3}$$

Assuming that the matrix B is invertible, (i.e., that $\det(B)\neq 0$), we introduce the function

$$f^{\Xi}_\hbar(x,z) \;=\; \exp\Big(\frac{\hbar^{\frac12}}{2}x^t 1 - \frac{\hbar^{\frac12}}{2}x^t B^{-1}\nu + \frac{\hbar}{8}f^t B^{-1}Af\Big)\prod_{i=1}^{N}\psi_\hbar(x_i,z_i) \in \mathbb{Q}(z)[x][\![\hbar^{\frac12}]\!]\,, \tag{4}$$

which is the product of one quantum dilogarithm per tetrahedron (each with its own integration variable), with some additional terms coming from the NZ-datum where $x = (x_1, ..., x_N)^t$ and $z = (z_1, ..., z_N)$.

The last ingredient is the formal Gaussian integration [5]

$$\langle f_\hbar(x,z)\rangle_{x,\Lambda} \;:=\; \exp\Bigg(\frac12\sum_{i,j=1}^{N}(\Lambda^{-1})_{i,j}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Bigg) f_\hbar(x,z)\Big|_{x=0} = \frac{\int e^{-\frac12 x^t\Lambda x} f_{\hbar,z}(x)\,dx}{\int e^{-\frac12 x^t\Lambda x}\,dx} \in \mathbb{Q}(z)[\![\hbar]\!] \tag{5}$$

of a power series $f_\hbar(x,z) \in \mathbb{Q}(z)[x][\![\hbar^{\frac12}]\!]$, with respect to an invertible matrix Λ. Assuming further that the symmetric matrix

$$\Lambda \;=\; -B^{-1}A + \mathrm{diag}(1/(1-z_j)) \tag{6}$$

is invertible, we define

$$\Phi^{\Xi}(\hbar) \;=\; \langle f_{\hbar}^{\Xi}(x,z)\rangle_{x,\Lambda} \in \mathbb{Q}(z)[\![\hbar]\!]\,. \tag{7}$$

Using a solution z of the Neumann–Zagier equations that corresponds to the discrete faithful representation $\rho^{\mathrm{geom}} : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ of a cusped hyperbolic 3-manifold M, one obtains a series $\Phi^{\Xi}(\hbar)$ with coefficients in the invariant trace field of M. Our main theorem is the following.

Theorem 1.1. $\Phi^{\Xi}(\hbar)$ *is a topological invariant of a cusped hyperbolic 3-manifold.*

A corollary of the above theorem is that every coefficient of the above series is a topological invariant of a cusped hyperbolic 3-manifold. These topological invariants have a geometric origin, since they take values in the invariant trace field of the manifold, and hence they are algebraic, but not (in general) rational, numbers. This series seems to come from hyperbolic geometry (though a definition of these invariants in terms of the hyperbolic geometry of the 3-manifold is not known), and perhaps not from enumerative quantum field theory (such as the Gromov–Witten or any of the 4-dimensional known theories). The behavior of these series under finite cyclic coverings of cusped hyperbolic 3-manifolds is given in [19], and the formulas presented there (e.g., for the coefficient of $\hbar^2$) point to unknown connections with the spectral theory of hyperbolic 3-manifolds.

The proof of our theorem combines the ideas of the topological invariance of the Andersen–Kashaev state-integrals [2] with those of the Aarhus integral [4]. We briefly recall that the building block for the state-integral is the Faddeev quantum dilogarithm, the state-integral is a finite-dimensional integral of a product of Faddeev quantum dilogarithm [15], one for each tetrahedron of an (ordered) ideal triangulation. Aside from elementary choices, two important parts in the proof of topological invariance of the state-integral is invariance under (a) the choice of a nondegenerate quad, and (b) 2–3 ordered Pachner moves, the latter connecting one ideal triangulation with another. In [2], (a) and (b) are dealt with a Fourier transform and a pentagon identity for the Faddeev quantum dilogarithm.

The power series $\Phi^{\Xi}(\hbar)$ is given instead by a formal Gaussian integral (as opposed to an integral over Euclidean space), where the building block is the formal power series $\psi_{\hbar}$ instead of the Faddeev quantum dilogarithm. The topological invariance of the $\Phi^{\Xi}(\hbar)$ under the choice of quad and under the 2–3 Pachner moves follows from a Fourier tranform identity and a pentagon identity for $\psi_{\hbar}$; see Theorems 3.4 and 3.6 below. Although these identities are, in a sense, limits of the corresponding identities for the Faddeev quantum dilogarithm (just as $\psi_{\hbar}$ is a limit of the Faddeev quantum dilogarithm), it would requires additional analytic work to do so, and instead we give algebraic proofs of theorems 3.4 and 3.6 using properties of the formal Gaussian integration, together with holonomic properties of the involved formal power series.

Having discussed the similarities between the proof of Theorem 1.1 and the corresponding theorem for the state-integral of [2], we now point out some differences. In [2], Andersen–Kashaev use ordered triangulations, and the state-integral is obtained by the push-forward of a distribution with variables at the faces and the tetrahedra of the ordered triangulation. Part of the distribution is a product of delta functions in linear forms of the face-variables. In our Theorem 1.1, and in the formal Gaussian integration, we carefully avoided the need to use delta functions, although such a reformulation of our results are possible, with additional effort.

We end this section by pointing out a wider context for the asymptotic series $\Phi^{\Xi}(\hbar)$ and Theorem 1.1. It was clear from [11] that a NZ-datum $\Xi = (A, B, \nu, z, f, f'')$ depends on two square matrices A and B such that AB^t is symmetric that may or may not come from topology, and that the series $\Phi^{\Xi}(\hbar)$ is defined under the assumption that $\det(B) \neq 0$ and $\det(\Lambda) \neq 0$. Doing so, the proof of Theorem 1.1 shows that the series $\Phi^{\Xi}(\hbar)$ is invariant under the moves that appear in [21, Sec.6]; see also Section 4.

2. Preliminaries

2.1. **The Faddeev quantum dilogarithm.** In this subsection, we recall some basic properties of the Faddeev quantum dilogarithm, which are motivations for Theorems 3.4 and 3.6 below. At the same time, we will ignore additional properties of the Faddeev quantum dilogarithm that play no role in our paper, such as the fact that it is a meromorphic function with precise zeros, poles and residues.

The Faddeev quantum dilogarithm [15] $\varphi := \Phi_{\mathsf{b}}$ satisfies a key integral pentagon identity [16]

$$e^{2\pi i x y}\widetilde{\varphi}(x)\widetilde{\varphi}(y) = \int_{\mathbb{R}} e^{\pi i z^2}\widetilde{\varphi}(x-z)\widetilde{\varphi}(z)\widetilde{\varphi}(y-z)\,\mathrm{d}z \tag{8}$$

where both sides are tempered distributions on $\mathbb{R}$ and $\widetilde{\varphi}$ denotes the distributional (inverse) Fourier transformation

$$\widetilde{\varphi}(x) := \int_{\mathbb{R}} e^{-2\pi i x y}\varphi(y)\,\mathrm{d}y\,. \tag{9}$$

It turns out that the inverse Fourier transform of $\varphi^{\pm 1}$ is expressed in terms of φ as given in [2, Sec.13.2]

$$\begin{aligned}\int_{\mathbb{R}} e^{-2\pi i x y}\varphi(y)\,\mathrm{d}y &= \zeta_8 (q/\widetilde{q})^{\frac{1}{24}} e^{-\pi i x^2}\varphi(-x+c_{\mathsf{b}})\\ \int_{\mathbb{R}} e^{-2\pi i x y}\varphi(y)^{-1}\,\mathrm{d}y &= \zeta_8^{-1} (\widetilde{q}/q)^{\frac{1}{24}} e^{\pi i x^2}\varphi(x-c_{\mathsf{b}})\end{aligned} \tag{10}$$

where $q = e^{2\pi i \mathsf{b}^2}$, $\widetilde{q} = e^{-2\pi i/\mathsf{b}^2}$, $c_{\mathsf{b}} = \frac{i}{2}(\mathsf{b}+\mathsf{b}^{-1})$ and $\zeta_8 = e^{2\pi i/8}$. The Faddeev quantum dilogarithm satisfies the inversion formula

$$\varphi(x)\varphi(-x) = (\widetilde{q}/q)^{\frac{1}{24}} e^{\pi i x^2} \tag{11}$$

see for example [2, App.A]. In a certain domain, the Faddeev quantum dilogarithm is given as a ratio of two Pochhammer symbols

$$\Phi_{\mathsf{b}}(x) \;=\; \frac{(-q^{\frac{1}{2}} e^{2\pi i \mathsf{b} x}; q)_\infty}{(-\widetilde{q}^{\frac{1}{2}} e^{2\pi i \mathsf{b}^{-1} x}; \widetilde{q})_\infty}\,, \tag{12}$$

from which it follows that its asymptotic expansion as $\mathsf{b} \to 0$ is given by replacing the denominator by 1 and the numerator by the asymptotic expansion of the Pochhammer symbol.

2.2. **Neumann–Zagier data.** In this section, we discuss in detail Neumann–Zagier data following [11]. We start with a 3-manifold M with torus boundary component (all manifolds and their triangulations will be oriented throughout the paper) equipped with a concrete oriented ideal triangulation, that is with a triangulation such that each tetrahedron Δ of $\mathcal{T}$ comes with a bijection of its vertices with those of the standard 3-simplex. (All triangulations that are used in the computer programs `SnapPy` [8] and `Regina` [7] are concrete).

Every concrete tetrahedron Δ has shape parameters (z, z', z'') assigned to pairs of opposite edges as in Figure 1, where the complex numbers $z' = 1/(1-z)$ and $z'' = 1 - 1/z$ satisfy the equations

$$z'' + z^{-1} = 1, \qquad zz'z'' = -1 \,. \tag{13}$$

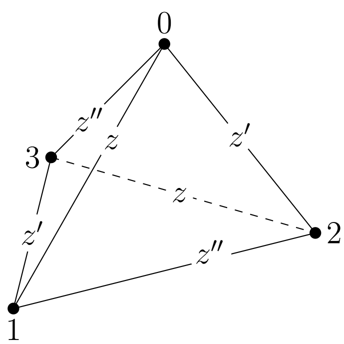

FIGURE 1. The shapes of an ideal tetrahedron.

If $\mathcal{T}$ is a triangulation as above, an Euler characteristic argument shows that the number of tetrahedra equals to the number of edges. Fix an ordering of the tetrahedra Δ_j for $j = 1, \dots, N$ and of the edges $e_1, \dots, e_N$ of $\mathcal{T}$, and assign a shape z_j to the tetrahedron Δ_j for $j = 1, \dots, N$. To describe the complete hyperbolic structure of M (when it exists) and its deformations, Thurston [34] introduced the gluing equations for the variables $z = (z_1, \dots, z_N)$ around each edge e_i, for $i = 1, \dots, N$. In logarithmic form, these equations have the form

$$\sum_{j=1}^{N} G_{i,j} \log z_j + G'_{i,j} \log z'_j + G''_{i,j} \log z''_j \;=\; 2\pi i, \qquad (i = 1, \dots, N) \tag{14}$$

where $G_{i,j}$ (and likewise for $G'_{i,j}$ and $G''_{i,j}$) denote the number of times that an edge of Δ_j labelled z_j winds around the edge e_i. Every peripheral curve γ in the boundary also gives rise to a gluing equation of the same form as (14), except the right hand side is 0 instead of $2\pi i$. Choosing a symplectic basis for $H_1(\partial M, \mathbb{Z})$, we can enhance the equations (14) by adding the peripheral equations

$$\sum_{j=1}^{N} G_{N+c,j} \log z_j + G'_{N+c,j} \log z'_j + G''_{N+c,j} \log z''_j \;=\; 0, \qquad (c = 1, 2) \,. \tag{15}$$

It turns out that the $(N{+}2) \times N$ matrices G, G' and G'' have both symmetry and redundancy. Any one of the edge equations is implies by the others, and we make a choice to remove one of them, and replace it by one peripheral equation for a fixed peripheral curve, resulting into $N \times N$ matrices $\mathbf{G}$, $\mathbf{G}'$ and $\mathbf{G}''$. Using the second Equation (13) in logarithmic form $\log z_j + \log z_j' + \log z_j'' = \pi i$, we can eliminate one of the three shapes of each tetrahedron (this is a choice of quad). For example, eliminating the shape z_j' for $j = 1, \dots, N$ then results into a system of equations

$$\sum_{j=1}^{N} A_{i,j} \log z_j + B_{i,j} \log z_j'' \;=\; \pi i \nu \qquad (i = 1, \dots, N) \tag{16}$$

where

$$A \;=\; \mathbf{G} - \mathbf{G}', \qquad B \;=\; \mathbf{G}'' - \mathbf{G}' \tag{17}$$

are the Neumann–Zagier matrices [29] and $\nu = (2, \dots, 2, 0)^t - \mathbf{G}'(1, \dots, 1, 1)^t \in \mathbb{Z}^N$. The Neumann–Zagier matrices $(A \,|\, B)$ have an important symplectic property, they are the upper part of a symplectic matrix over $\mathbb{Z}[1/2]$ (and even a symplectic matrix over $\mathbb{Z}$ if one divides the peripheral gluing equation by 2 while keeping the integrality of its coefficients) [29]. It follows that AB^t is symmetric, that $(A \,|\, B)$ has full rank, and that one can always choose a quad such that B is invertible–for this see [11, Lem.A.3].

A further ingredient of a Neumann–Zagier datum is a flattening, that is a triple (f, f', f'') of vectors in $\mathbb{Z}^N$ that satisfy the conditions

$$f + f' + f'' = (1, \dots, 1)^t, \qquad \mathbf{G} f + \mathbf{G}' f' + \mathbf{G}'' f'' = (2, \dots, 2, 0)^t \,. \tag{18}$$

Using our choice of quad, we can eliminate f' and thus obtain a pair (f, f'') of vectors in $\mathbb{Z}^N$ that satisfy the condition

$$Af + Bf'' = \nu \,. \tag{19}$$

This defines all the terms that appear in a NZ-datum $\Xi = (A, B, \nu, z, f, f'')$. The definition of the series $\Phi^{\Xi}(\hbar)$ requires a nondegenerate NZ datum Ξ, that is one that satisfies the condition $\det(B) \neq 0$, which as we discussed above, can always be achieved, as well as the condition $\det(\Lambda) \neq 0$. We discuss this choice next, and connect it to the geometric representation of a hyperbolic 3-manifold M.

We end this section with a comment regarding Neumann–Zagier data of 3-manifolds with several (as opposed to one) torus boundary components. Their ideal triangulations with equal number N of tetrahedra as edges and the edge gluing equations have the same shape (14) as above. If r denotes the number of boundary components, then there are $2r$ peripheral equations (15) and after a choice of one peripheral curve per boundary component, this leads to $(N + r) \times N$ matrices G, G' and G''. The edge gluing equations have redundancy, and although it is not true that we can remove any r of them, it is shown in [18, Sec.4.6] that one can remove r of them and and replace them by r peripheral equations so as to obtain $N \times N$ matrices $\mathbf{G}$, $\mathbf{G}'$ and $\mathbf{G}''$ such that the corresponding matrices $(A \,|\, B)$ defined in (17) have the same symplectic properties as in the case of $r = 1$. Moreover, any two choices of removal of the r edge equations are related to each other by an invertible matrix in $\mathrm{GL}_r(\mathbb{Z})$. Finally, flattenings satisfy Equations (18) where the right hand side of the second equation is the vector $(2, \dots, 2, 0, \dots, 0)^t \in \mathbb{Z}^{N+r}$ with the first N coordinates equal to 2 and the remaining r coordinates equal to zero.

For simplicity in the presentation (related to the choice of peripheral curves and the flattenings), we will give the proof of Theorem 1.1 assuming that the 3-manifold M has one cusp. The proof applies verbatim to the general case of arbitrary number of cusps.

2.3. **Geometric aspects.** In this section, we discuss some geometric aspects of Theorem 1.1 related to the choice of the shapes z in a Neumann–Zagier datum. Let us fix an ideal triangulation $\mathcal{T}$. A solution $z \in (\mathbb{C} \setminus \{0,1\})^N$ to the Neumann–Zagier equations gives rise, via a developing map, to a representation $\rho_z : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$. For a detailed discussion, see the appendix of [6]. However, not every representation $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ is "seen" by $\mathcal{T}$, that is, is in the image of the map $z \to \rho_z$. What's more, if ρ is in the image of the above map and we do a 2–3 Pachner move on the triangulation, it may no longer be in the image of the map corresponding to the new triangulation. The reason is that the shapes of the two triangulations are related by a rational map, which may send a complex number different from 0 and 1 to 0, 1 or ∞. To phrase the problem differently, every two ideal triangulations (each with at least two tetrahedra, as we will always assume) of a 3-manifold with non-empty boundary are related by a sequence of 2–3 Pachner moves [28, 32]. However, it is not known that the set of ideal triangulations that see the discrete faithful representation $\rho^{\mathrm{geom}} : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ is connected under 2–3 Pachner moves, nor is it known whether the set of nondegenerate NZ data is connected under 2–3 Pachner moves.

A solution to these issues was found in [11], and this was used to prove the topological invariance of the 1-loop function, and was also used in [18] to prove the topological invariance of the 3D-index. Let us recall the geometric details here. Every cusped hyperbolic 3-manifold M (complete, finite volume) has a canonical ideal cell decomposition whose cells are 3-dimensional convex ideal polytopes given by Epstein–Penner [13]. It is easy to see that every convex ideal polyhedron can be triangulated into ideal tetrahedra by connecting an ideal vertex to all other ideal vertices (thus reducing the problem to the ideal cone of an ideal polygon), and then triangulating further the ideal polygon into ideal triangles. Doing so, the triangulation of the common faces of the 3-dimensional convex ideal polytopes may not match, in which case one can pass from one triangulation of a polygonal face to another by adding flat tetrahedra.

The question is whether every two such triangulations are related by a sequence of 2–3 moves. This is a combinatorial problem of convex geometry, which we summarize below. For a detailed discussion, the reader may consult the book [9] and references therein.

Fix a convex polytope P in $\mathbb{R}^d$. One can consider the set of triangulations of P. When $d = 2$, P is a polygon and it is known that every two triangulations are related by a sequence of flips. For general d, flips are replaced by *geometric bistellar moves*. When $d \geq 5$, it is known that the graph of triangulations (with edges given by geometric bistellar flips) is not connected, and has isolated vertices. For $d = 3$, it is not known whether the graph is connected.

The situation is much better when one considers *regular triangulations* of P. In that case, the corresponding graph of regular triangulations is connected and, in fact, it is the edge set of the *secondary polytope* of P. When $d = 3$ and P is convex and in general position, then the only geometric bistellar move is the 2–3 move where the added edge that appears in the move is an edge that connects two vertices of P. When $d = 3$ and P is not in general position, the same conclusion holds as long as one allows for tetrahedra that are flat, i.e., lie on a 2-dimensional plane.

Returning to the Epstein-Penner ideal cell decomposition, let $\mathcal{T}^{\mathrm{EP}}$ denote a regular ideal triangulations of the canonical ideal cell decomposition of M. (For a detailed discussion of this set, see also [18, Sec.6]). The set $\{\mathcal{T}^{\mathrm{EP}}\}$ of regular ideal triangulations is connected by 2–3 Pachner moves. Moreover, such triangulations see ρ^{geom} since by the geometric construction, the shapes are always nondegenerate, i.e., not equal to 0 or 1 and in fact always have nonnegative (though sometimes zero) imaginary part.

Finally, we need to show that $\det(\Lambda)$ is nonzero. This follows from the fact that $\det(B)\det(\Lambda) = \det(-A + B\mathrm{diag}(1/(1-z_j))$ equals (up to multiplication by a monomial in z and z'') to the 1-loop invariant of [11, Sec.1.3]. The nonvanishing of the latter follows from Thurston's hyperbolic Dehn surgery theorem [34], which implies that $\rho^{\mathrm{geom}} \in X_M^{\mathrm{geom}} \cup P_M$ is an isolated smooth point of the geometric component X_M^{geom} of the $\mathrm{PSL}_2(\mathbb{C})$-character variety of M, intersected with the locus P_M of boundary-parabolic $\mathrm{PSL}_2(\mathbb{C})$-representations, i.e., representations ρ satisfying $\mathrm{tr}(\rho(\gamma))^2 = 4$ for all peripheral elements $\gamma \in \pi_1(M)$. Since the 1-loop invariant is the determinant of the Hessian of the defining NZ equations of ρ^{geom}, it follows that the 1-loop invariant is nonzero.

3. Formal Gaussian integration

3.1. **Basics on formal Gaussian integration.** In this section, we review the basic properties of formal Gaussian integration, which is a combinatorial analogue of integration of analytic functions. This theory was introduced and studied in detail in [4], where it was used to define a universal perturbative finite type invariant of rational homology 3-spheres, and to identify it with the trivial connection contribution of Chern–Simons perturbation theory.

As a warm-up, the formal Gaussian integral of a monomial x^n in one variable with respect to the quadratic form $\lambda \neq 0$ is defined by

$$\langle x^n \rangle_{x,\lambda} \;=\; \begin{cases} \lambda^{-n/2}(n-1)!! & n \text{ even} \\ 0 & n \text{ odd}\,. \end{cases} \tag{20}$$

When $\lambda > 0$, then the above bracket equals to a normalized Gaussian integral

$$\langle x^n \rangle_{x,\lambda} \;=\; \frac{\int_{\mathbb{R}} e^{-\frac{1}{2}\lambda x^2} x^n \, dx}{\int_{\mathbb{R}} e^{-\frac{1}{2}\lambda x^2} \, dx}\,, \tag{21}$$

explaining the naming of formal Gaussian integration. The formal Gaussian integration can be extended by linearity to $\langle f(x)\rangle_{x,\lambda}$ for any polynomial $f(x)$, or even further to a power series $f(x) = \sum_{n\geq 0} a_n x^n$ whose coefficients tend to zero in a local ring (such as the ring $\mathbb{Q}[\![\hbar^{\frac{1}{2}}]\!]$).

The formal Gaussian integral (20) has a multivariable extension given in Equation (5) where x is a vector of variables and Λ is an invertible matrix over a field matching the size of x. Throughout this paper, the entries of Λ are elements of the field $\mathbb{Q}(z)$ where z is a vector of variables, the integrable functions $f_\hbar(x,z)$ that we apply the formal Gaussian integration with respect to x with be elements of $\mathbb{Q}(z)[x][\![\hbar^{\frac{1}{2}}]\!]$, and the result of formal Gaussian integration will be an element of the ring $\mathbb{Q}(z)[\![\hbar]\!]$ or $\mathbb{Q}(z)[x][\![\hbar^{\frac{1}{2}}]\!]$. When we specialize to the case of a vector of algebraic numbers z, then $\mathbb{Q}(z)$ defines the corresponding number field.

Just like integration of sufficiently smooth functions satisfies certain invariance properties (such as change of variables, iterated integration, and even integration by parts [4]), so does formal Gaussian integration. The corresponding identities of formal Gaussian integration are combinatorial statements about polynomials or rational functions and often follow from the corresponding statements of integration of functions.

We now give some elementary properties of formal Gaussian integration that we will use in our paper.

Lemma 3.1. (a) For all invertible matrices Λ and P over $\mathbb{Q}(z)$, we have

$$\langle f_\hbar(Px,z)\rangle_{x,P^t\Lambda P} \;=\; \langle f_\hbar(x,z)\rangle_{x,\Lambda}\,. \tag{22}$$

(b) For all invertible matrices Λ and vectors G over $\mathbb{Q}(z)$, we have

$$\langle \exp(-G^t\Lambda x\hbar^{\frac{1}{2}})f_\hbar(x+G\hbar^{\frac{1}{2}},z)\rangle_{x,\Lambda} \;=\; \exp\left(\frac{G^t\Lambda G}{2}\hbar\right)\langle f_\hbar(x,z)\rangle_{x,\Lambda} \tag{23}$$

(c) If $\Lambda=\left(\begin{smallmatrix} A & B\\ B^t & C\end{smallmatrix}\right)$, and Λ and A invertible, then for any F, we have

$$\Big\langle \exp(Fx'\hbar^{\frac{1}{2}})f_\hbar(x'',z)\Big\rangle_{x,\Lambda} \;=\; \exp\left(\frac{FA^{-1}F^t}{2}\hbar\right)\Big\langle \exp(-FA^{-1}Bx''\hbar^{\frac{1}{2}})f_\hbar(x'',z)\Big\rangle_{x'',C-B^tA^{-1}B}. \tag{24}$$

Proof. Part (a) follows from the fact the integration is unchanged under a linear change of variables.

Part (b) follows from the fact the integration is unchanged under an affine change of variables.

Part (c) follows from Fubini's theorem [4, Prop.2.13], combined with Equation (23). ☐

The next lemma, which will be important in the application of q-holonomic methods in Section 3.2 and in the proofs of Theorems 3.4 and 3.6 below, concerns the behavior of formal Gaussian integration when $z=(z_1,\dots,z_r)$ is shifted to $e^{\varepsilon\hbar}z:=(e^{\varepsilon_1\hbar}z_1,\dots,e^{\varepsilon_r\hbar}z_r)$ for integers ε_j.

Lemma 3.2. For all invertible matrices $\Lambda(z)$ over $\mathbb{Q}(z)$ and all integrable functions $f_\hbar$, we have

$$\langle f_\hbar(x,z)\rangle_{x,\Lambda(z)}|_{z\mapsto e^{\varepsilon\hbar}z} \;=\; \sqrt{\frac{\det\Lambda(e^{\varepsilon\hbar}z)}{\det\Lambda(z)}}\Big\langle \exp\left(\frac{x^t\big(\Lambda(z)-\Lambda(e^{\varepsilon\hbar}z)\big)x}{2}\right)f_\hbar(x,e^{\varepsilon\hbar}z)\Big\rangle_{x,\Lambda(z)}. \tag{25}$$

Proof. The lemma follows from recentering the Gaussian after multiplying z by $e^{\varepsilon\hbar}$. ☐

3.2. **q–holonomic aspects of $\psi_\hbar$.** It is well-known that identities of holonomic functions can be proven algorithmically [36, 31]. We will use these ideas, adapted to our needs, to prove fundamental identities between certain Gaussian integrals involving the building block $\exp(\psi_\hbar)$ of our series $\Phi^\Xi(\hbar)$. Since $\psi_\hbar$ is related to the infinite Pochhammer symbol given in Equation (2), its functional equations will be of fundamental importance. From its definition, it is clear that the infinite Pochhammer symbol satisfies a simple first order linear q–difference equation

$$(x;q)_\infty \;=\; (1-x)(qx;q)_\infty\,. \tag{26}$$

To convert this into an identity of formal $\hbar$-series where $q=e^\hbar$, we use the fact that there is an action of the quantum plane (also known as q–Weyl [14, Ex. 1.7] algebra) on the space $\mathbb{Q}(z)[x][\![\hbar^{\frac{1}{2}}]\!]$ of integrable functions by an action on the z-variable given by

$$(Lf_\hbar)(x,z) \;=\; f_\hbar(x,e^\hbar z), \qquad (Mf_\hbar)(x,z) \;=\; e^\hbar f_\hbar(x,e^\hbar z) \tag{27}$$

where $LM=qML$. The next lemma asserts that the completion $\widehat{\psi}_\hbar(x,z)$

$$\begin{aligned}\widehat{\psi}_\hbar(x,z) \;:=&\; \exp\Big(-\frac{\mathrm{Li}_2(z)}{\hbar}-\frac{\mathrm{Li}_1(z)x}{\hbar^{\frac{1}{2}}}+\frac{1}{2}\,\mathrm{Li}_1(z)-\frac{\mathrm{Li}_0(z)x^2}{2}\Big)\psi_\hbar(x,z)\\ =&\; \exp\Big(-\sum_{k,\ell\in\mathbb{Z}_{\geq 0}}\frac{B_k\,x^\ell\,\hbar^{k+\frac{\ell}{2}-1}}{\ell!\,k!}\,\mathrm{Li}_{2-k-\ell}(z)\Big)\\ \in&\; \exp\Big(-\frac{\mathrm{Li}_2(z)}{\hbar}-\frac{\mathrm{Li}_1(z)x}{\hbar^{\frac{1}{2}}}+\frac{1}{2}\,\mathrm{Li}_1(z)-\frac{\mathrm{Li}_0(z)x^2}{2}\Big)(1+\hbar^{\frac{1}{2}}\mathbb{Q}(z)[x][\![\hbar^{\frac{1}{2}}]\!])\end{aligned} \tag{28}$$

of $\psi_\hbar(x,z)$, as well as $\psi_\hbar(x,z)$ itself, satisfy a corresponding first order linear q–difference equation, albeit with complicated coefficients.

Lemma 3.3. (a) We have:

$$\widehat{\psi}_\hbar(x,e^\hbar z) \;=\; (1-ze^{x\hbar^{1/2}})\,\widehat{\psi}_\hbar(x,z)\,. \tag{29}$$

(b) We have:

$$\begin{aligned}\psi_\hbar(x,e^\hbar z) \;=&\; (1-ze^{x\hbar^{1/2}})\,\psi_\hbar(x,z)\sqrt{\frac{1-e^\hbar z}{1-z}}\\ &\times\exp\Big(\frac{\mathrm{Li}_2(e^\hbar z)}{\hbar}-\frac{\mathrm{Li}_2(z)}{\hbar}+\frac{\mathrm{Li}_1(e^\hbar z)x}{\hbar^{\frac{1}{2}}}-\frac{\mathrm{Li}_1(z)x}{\hbar^{\frac{1}{2}}}+\frac{1}{2}\,\mathrm{Li}_0(e^\hbar z)x^2-\frac{1}{2}\,\mathrm{Li}_0(z)x^2\Big)\,.\end{aligned} \tag{30}$$

Note that the identity (29) takes place in a larger ring, which includes the symbols $\exp(\frac{\mathrm{Li}_2(z)}{\hbar})$, $\exp(\frac{\mathrm{Li}_1(z)x}{\hbar^{\frac{1}{2}}})$, $\exp(\frac{\mathrm{Li}_0(z)x^2}{2})$ and $\exp(\frac{\mathrm{Li}_0(z)x^2}{2})$ which one can adjoin in the differential field $\mathbb{Q}(z)((\hbar))$ as is standard in differential Galois theory of linear differential equations [35]. The symbols $\mathrm{Li}_k(z)$ for $k=0,1,2$ can be interpreted as normalized solutions to the linear differential equations $z\frac{d}{dz}\mathrm{Li}_k(z)=\mathrm{Li}_{k-1}(z)$ with $\mathrm{Li}_0(z)=z/(1-z)$ and satisfy the usual properties $\mathrm{Li}_k(e^\hbar z)=\sum_{r=0}^\infty\frac{1}{r!}\,\mathrm{Li}_{k-r}(z)\hbar^r$. On the other hand, the coefficients in identity (30) are elements of the ring $\mathbb{Q}(z)[x][\![\hbar^{\frac{1}{2}}]\!]$.

Proof. Part (a) can be proved directly using the facts that for the Bernoulli polynomials $B_r(x)$ we have $B_r(1)=B_r+\delta_{r,1}$ and $B_r(x)=\sum_{k=0}^r\binom{r}{k}B_{n-k}x^k$. Applying these identities we find that

$$\begin{aligned}
&\exp\Big(-\sum_{k,\ell\in\mathbb{Z}_{\geq 0}} \frac{B_k x^\ell \hbar^{k+\frac{\ell}{2}-1}}{\ell!k!}\operatorname{Li}_{2-k-\ell}(ze^{\hbar})\Big) \\
&= \exp\Big(-\sum_{r,\ell\in\mathbb{Z}_{\geq 0}} \sum_{k=0}^{r} B_k \binom{r}{k} \frac{x^\ell \hbar^{r+\frac{\ell}{2}-1}}{\ell!r!}\operatorname{Li}_{2-r-\ell}(z)\Big) \\
&= \exp\Big(-\sum_{r,\ell\in\mathbb{Z}_{\geq 0}} \frac{B_r(1) x^\ell \hbar^{r+\frac{\ell}{2}-1}}{\ell!r!}\operatorname{Li}_{2-r-\ell}(z)\Big) \\
&= \exp\Big(-\operatorname{Li}_1(ze^{x\hbar^{1/2}}) - \sum_{r,\ell\in\mathbb{Z}_{\geq 0}} \frac{B_r x^\ell \hbar^{r+\frac{\ell}{2}-1}}{\ell!r!}\operatorname{Li}_{2-r-\ell}(z)\Big) \\
&= (1-ze^{x\hbar^{1/2}})\exp\Big(-\sum_{r,\ell\in\mathbb{Z}_{\geq 0}} \frac{B_r x^\ell \hbar^{r+\frac{\ell}{2}-1}}{\ell!r!}\operatorname{Li}_{2-r-\ell}(z)\Big)\,.
\end{aligned} \tag{31}$$

Part (b) follows from (a), using Equation (28) and expanding in $\hbar$ to find that

$$\sqrt{\frac{1-e^{\hbar}z}{1-z}}\exp\Big(\frac{\operatorname{Li}_2(e^{\hbar}z)}{\hbar} - \frac{\operatorname{Li}_2(z)}{\hbar} + \frac{\operatorname{Li}_1(e^{\hbar}z)x}{\hbar^{\frac{1}{2}}} - \frac{\operatorname{Li}_1(z)x}{\hbar^{\frac{1}{2}}} + \frac{1}{2}\operatorname{Li}_0(e^{\hbar}z)x^2 - \frac{1}{2}\operatorname{Li}_0(z)x^2\Big) \in \mathbb{Q}(z)[x][\![\hbar^{\frac{1}{2}}]\!]\,. \tag{32}$$

□

We end this section by discussing a useful relation between $\psi_\hbar(x,z)$ and its specialization at $x=0$. It is easy to see that the completion $\widehat{\psi}_\hbar(x,z)$ of $\psi_\hbar(x,z)$ is regular at $x=0$ and satisfies

$$\widehat{\psi}_\hbar(x,z) \;=\; \widehat{\psi}_\hbar(0, ze^{x\hbar^{1/2}})\,. \tag{33}$$

This implies a corresponding statement

$$\psi_\hbar(x,z) \;=\; \psi_\hbar\big(0, ze^{x\hbar^{1/2}}\big)C_\hbar(x,z), \tag{34}$$

for $\psi_\hbar$ where

$$\begin{aligned}
C_\hbar(x,z) \;&=\; \exp\Big(-\sum_{\ell\geq 3}\frac{\hbar^{\frac{\ell}{2}-1}}{\ell!}\operatorname{Li}_{2-\ell}(z)x^\ell + \frac{1}{2}\sum_{\ell\geq 1}\frac{h^{\frac{\ell}{2}}}{\ell!}\operatorname{Li}_{1-\ell}(z)x^\ell\Big) \\
&=\; \exp\Big(-\frac{\operatorname{Li}_2(ze^{x\hbar^{\frac{1}{2}}})}{\hbar} + \frac{\operatorname{Li}_2(z)}{\hbar} - \frac{\log(1-z)}{\hbar^{\frac{1}{2}}}x + \frac{z}{1-z}\frac{x^2}{2} \\
&\qquad - \frac{1}{2}(\log(1-ze^{z\hbar^{\frac{1}{2}}}) - \log(1-z))\Big)\,.
\end{aligned} \tag{35}$$

3.3. **Fourier transform of $\psi_\hbar$.** In this section, we prove two functional identities for $\psi_\hbar(\bullet,z)$, $\psi_\hbar(\bullet,z')$ and $\psi_\hbar(\bullet,z'')$ where $z' := 1/(1-z)$ and $z'' := 1-1/z$ are related to the $\mathbb{Z}/3\mathbb{Z}$-symmetry of the shapes of a tetrahedron, and reflect the $\mathbb{Z}/3\mathbb{Z}$-symmetry of the dilogarithm function.

The next theorem is a formal Gaussian integration version of the Fourier transform of the Faddeev quantum dilogarithm given in Equation (9). The proof, however, does not follow from the distributional identity (9), and instead uses q–holonomic ideas and the basics of formal Gaussian integration.

This theorem and its Corollary 3.5 are used in Section 5 to show that the power series $\Phi^{\Xi}(\hbar)$ are independent of the choice of a nondegenerate quad.

Theorem 3.4. *We have:*

$$\psi_\hbar(x,z) \;=\; e^{-\frac{\hbar}{24}}\Big\langle \exp\Big(\Big(y+\frac{xz}{1-z}\Big)\frac{\hbar^{\frac{1}{2}}}{2}\Big)\,\psi_\hbar\Big(y+\frac{xz}{1-z},\frac{1}{1-z}\Big)\Big\rangle_{y,1-z^{-1}} \tag{36}$$

in the ring $\mathbb{Q}(z)[x][\![\hbar^{\frac{1}{2}}]\!]$.

In fact, both sides of of Equation (36) lie in the subring $\mathbb{Q}[z^{\pm},(1-z)^{-1}][x][\![\hbar^{\frac{1}{2}}]\!]$.

Proof. It it clear from the definition that both sides of Equation (36) are elements of the ring $\mathbb{Q}(x)[z][\![\hbar]\!]$. First, we will prove the specialization of Equation (36) when $x=0$, i.e., we will show that

$$\psi_\hbar(0,z) \;=\; e^{-\frac{\hbar}{24}}\Big\langle \exp\Big(\frac{y}{2}\hbar^{\frac{1}{2}}\Big)\,\psi_\hbar\Big(y,\frac{1}{1-z}\Big)\Big\rangle_{y,1-z^{-1}} \in \mathbb{Q}(z)[[\hbar]]\,. \tag{37}$$

To prove this, we will combine q-holonomic ideas with formal Gaussian integration. From Equation (29), we see that the left hand side of Equation (37) multiplied by $\frac{\exp(-\mathrm{Li}_2(z)\hbar^{-1})}{\sqrt{1-z}}$ satisfies a simple first order q–difference equation. We want to show the same for the right hand side. To do this, consider the function

$$I_{m,\hbar}(w,z) \;=\; w^m e^{-\frac{\hbar}{24}+\frac{\pi^2}{6\hbar}}\exp\Big(\log(w)\Big(-\frac{\log(w)}{2\hbar}+\frac{\pi i}{\hbar}-\frac{\log(z)}{\hbar}+\frac{1}{2}\Big)\Big)\,\widehat{\psi}_\hbar(0,w) \tag{38}$$

which is an element of a larger ring discussed in relation to $\widehat{\psi}_\hbar$ of Equation (28). Using Equation (34) and the symmetry

$$\mathrm{Li}_2(z) \;=\; \mathrm{Li}_2\Big(\frac{1}{1-z}\Big)-\frac{1}{2}\log(1-z)^2+\pi i\log(1-z)-\log(z)\log(1-z)-\frac{\pi^2}{6} \tag{39}$$

of the dilogarithm [39], it is easy to see that the right hand side of Equation (37) is given by

$$\Big\langle \sqrt{-z}\exp\Big(\frac{\mathrm{Li}_2(z)}{\hbar}+(1-z^{-1})\frac{w^2}{2}\Big)I_{0,\hbar}\Big(\frac{1}{1-z}e^{w\hbar^{1/2}},z\Big)\Big\rangle_{w,1-z^{-1}}\,. \tag{40}$$

The function $I_{m,\hbar}$ satisfies the linear q–difference equations

$$\begin{aligned} I_{m,\hbar}(e^{\hbar}w,z) &= -e^{m\hbar}w^{-1}z^{-1}(1-w)I_{m,\hbar}(w,z)\,,\\ I_{m,\hbar}(w,e^{\hbar}z) &= w^{-1}I_{m,\hbar}(w,z)\,,\\ I_{m,\hbar}(w,z) &= wI_{m,\hbar}(w,z)\,, \end{aligned} \tag{41}$$

which imply that

$$(1-e^{-m\hbar}z)I_{m,\hbar}\Big(\frac{1}{1-z}e^{w\hbar^{1/2}+\hbar},z\Big) \;=\; I_m\Big(\frac{1}{1-z}e^{w\hbar^{1/2}},e^{\hbar}z\Big). \tag{42}$$

If we multiply both sides of (42) with the factors from Equation (40) and take the bracket of both sides and apply Lemma 3.1 and Lemma 3.2 to change coordinates and the Gaussian, we find that when $m=0$ both sides of Equation (37) satisfy the same q–difference equation (29).

Moreover, it is easy to see that both sides of Equation (40) are power series in $\hbar$ with coefficients rational functions of z of nonpositive degree, thus we can work with the ring $\mathbb{Q}[\![z^{-1}]\!][\![\hbar]\!]$ instead. In this case, the Newton polygon of this first order linear q-difference equation implies that it has a unique solution in $\mathbb{Q}[\![z^{-1}]\!][\![\hbar]\!]$ determined by its value at $z = \infty$. Therefore, the identity in Equation (37) follows from its specialization at $z = \infty$. Since

$$\begin{aligned} \psi_\hbar(0,\infty) \;&=\; \exp\left(\frac{\hbar}{12}\right), \\ e^{\frac{-\hbar}{24}}\left\langle \exp\left(\frac{w}{2}\hbar^{\frac{1}{2}}\right)\psi_\hbar\left(w,0\right)\right\rangle_{w,1} \;&=\; e^{\frac{-\hbar}{24}}\left\langle \exp\left(\frac{w}{2}\hbar^{\frac{1}{2}}\right)\right\rangle_{w,1} \;=\; \exp\left(\frac{\hbar}{12}\right), \end{aligned} \tag{43}$$

this completes the proof of Equation (37).

Going back to the general case of z, Equation (36) follows from Equation (34) together with (37) using Lemma 3.2 to shift the Gaussian and Lemma 3.1 to change integration variable via the transformation

$$w \mapsto w - \frac{1}{\hbar^{\frac{1}{2}}}\log\left(\frac{1-z}{1-ze^{x\hbar^{1/2}}}\right) - \frac{x}{1-z^{-1}}\,. \tag{44}$$

The detailed calculation is given in Appendix A. □

Theorem 3.4 implies the following identity relating $\psi_\hbar(\bullet, z)$ and $\psi_\hbar(\bullet, z'')$.

Corollary 3.5. We have:

$$\psi_\hbar(x,z) \;=\; e^{\frac{\hbar}{24}}\Big\langle \exp\left(-\frac{x}{2}\hbar^{\frac{1}{2}}\right)\psi_\hbar\Big(y-\frac{x}{1-z}, 1-z^{-1}\Big)\Big\rangle_{y,z-1} \tag{45}$$

in the ring $\mathbb{Q}(z)[x][\![\hbar^{\frac{1}{2}}]\!]$.

Proof. Apply Equation (36) to the $\psi_\hbar$ that appears on the right hand side of the same Equation (36). Then apply a change of variables and Fubini's theorem of Lemma 3.1 to calculate the bracket for the variable that doesn't appear in the argument of the remaining $\psi_\hbar$. □

3.4. **Pentagon identity for $\psi_\hbar$.** In this section, we prove a pentagon identity for the functions $\psi_\hbar(\bullet, z)$ where z takes the five values

$$[z_1]+[z_2] \mapsto [z_1z_0^{-1}]+[z_0]+[z_2z_0^{-1}], \qquad z_0 \;=\; z_1+z_2-z_1z_2 \tag{46}$$

what appear in the 5-term relation for the dilogarithm. The next theorem is a formal Gaussian integration version of the pentagon identity (8) of the Faddeev quantum dilogarithm and will be used in Section 6 to prove that the series $\Phi^{\Xi}(\hbar)$ is independent of 2–3 Pachner moves. Let us denote

$$\delta \;=\; 2+\mathrm{Li}_0(z_1z_0^{-1})+\mathrm{Li}_0(z_0)+\mathrm{Li}_0(z_2z_0^{-1}) \;=\; \frac{(z_1+z_2-z_1z_2)^2}{z_1z_2(1-z_1)(1-z_2)}\,. \tag{47}$$

Theorem 3.6. *We have:*

$$\begin{aligned} \psi_\hbar(x,z_1)\,\psi_\hbar(y,z_2) \;&=\; e^{-\frac{\hbar}{24}}\Big\langle\psi_\hbar\Big(-w-y+\frac{xz_2+yz_1}{z_0}, z_1z_0^{-1}\Big) \\ \psi_\hbar\Big(w+x+y-\frac{xz_2+yz_1}{z_0}, z_0\Big)&\psi_\hbar\Big(-w-x+\frac{xz_2+yz_1}{z_0}, z_2z_0^{-1}\Big)\Big\rangle_{w,\delta} \end{aligned} \tag{48}$$

in the ring $\mathbb{Q}(z_1, z_2)[x, y][\![\hbar^{\frac{1}{2}}]\!]$.

Proof. It is clear from the definition that both sides of Equation (48) are elements of the ring $\mathbb{Q}(z_1, z_2)[x, y][\![\hbar^{\frac{1}{2}}]\!]$. To prove this, we follow the same approach we used to prove Theorem 3.4. First, we prove the identity (48) when $u = v = 0$, i.e., we will show that

$$\psi_\hbar(0, z_1)\, \psi_\hbar(0, z_2) \;=\; e^{-\frac{\hbar}{24}} \big\langle \psi_\hbar(-w, z_1 z_0^{-1})\, \psi_\hbar(w, z_0)\, \psi_\hbar(-w, z_2 z_0^{-1}) \big\rangle_{w,\delta} \tag{49}$$

in the ring $\mathbb{Q}(z_1, z_2)[\![\hbar]\!]$ and to prove this we will again use q–holonomic methods. To do so, consider the function

$$\begin{aligned} I_{m,\hbar}(z_1, z_2, z) \;&=\; e^{\frac{\pi^2}{6\hbar} - \frac{\hbar}{24}} z^m\, \widehat{\psi}_\hbar(0, z_1 z^{-1})\, \widehat{\psi}_\hbar(0, z)\, \widehat{\psi}_\hbar(0, z_2 z^{-1}) \\ &\times \exp\Big(- \frac{\log(z_1)\log(z_2)}{\hbar} + \frac{\log(z)\log(z_1)}{\hbar} + \frac{\log(z)\log(z_2)}{\hbar} - \frac{\log(z)^2}{\hbar} \Big)\,, \end{aligned} \tag{50}$$

which is again an element of a larger ring discussed in relation to $\widehat{\psi}_\hbar$ of Equation (28). Using Equation (34) and the five term relation for the dilogarithm [39], it is easy to see that the right hand side of Equation (49) is given by

$$\Big\langle \frac{\sqrt{z_1 z_2}(1 - z_1)(1 - z_2)}{z_1 + z_2 - z_1 z_2} \exp\Big(\frac{\mathrm{Li}_2(z_1)}{\hbar} + \frac{\mathrm{Li}_2(z_2)}{\hbar} + \frac{\delta}{2} z^2\Big) I_{0,\hbar}(z_1, z_2, z_0 e^{w\hbar^{1/2}}) \Big\rangle_{w,\delta}\,. \tag{51}$$

The function $I_{m,\hbar}$ satisfies the system of linear q–difference equations

$$\begin{aligned} I_{m,\hbar}(e^\hbar z_1, z_2, z) \;&=\; z\, z_2^{-1}(1 - z_1 z^{-1})\, I_{m,\hbar}(z_1, z_2, z)\,, \\ I_{m,\hbar}(z_1, e^\hbar z_2, z) \;&=\; z\, z_1^{-1}(1 - z_2 z^{-1})\, I_{m,\hbar}(z_1, z_2, z)\,, \\ I_{m,\hbar}(z_1, z_2, e^\hbar z) \;&=\; (1 - z)\, z_1 z_2 z^{-2} e^{(m-1)\hbar} I_{m,\hbar}(z_1, z_2, z)\,, \\ I_{m+1,\hbar}(z_1, z_2, z) \;&=\; z\, I_{m,\hbar}(z_1, z_2, z)\,, \end{aligned} \tag{52}$$

which can be derived from Equation (29). In fact, the function $I_{m,\hbar}$ is holonomic of rank 1, a fact that we will not use explicitly [31]. Therefore, we see that

$$\begin{aligned} &I_{m,\hbar}(e^\hbar z_1, z_2, z) \;=\; z_2^{-1} I_{m+1,\hbar}(z_1, z_2, z) - z_1 z_2^{-1} I_{m,\hbar}(z_1, z_2, z)\,, \\ &I_{m,\hbar}(e^\hbar z_1, z_2, (e^\hbar z_1 + z_2 - e^\hbar z_1 z_2) e^{w\hbar^{1/2}}) \\ &\;=\; z_2^{-1} I_{m+1,\hbar}(z_1, z_2, (e^\hbar z_1 + z_2 - e^\hbar z_1 z_2) e^{w\hbar^{1/2}}) - z_1 z_2^{-1} I_{m,\hbar}(z_1, z_2, (e^\hbar z_1 + z_2 - e^\hbar z_1 z_2) e^{w\hbar^{1/2}})\,, \\ &I_{m,\hbar}(z_1, z_2, (z_1 + z_2 - z_1 z_2) e^{z\hbar^{1/2} + \hbar}) \\ &\;=\; z_1 z_2 e^{(m-1)\hbar} I_{m-2,\hbar}(z_1, z_2, (z_1 + z_2 - z_1 z_2) e^{z\hbar^{1/2}}) - z_1 z_2 e^{(m-1)\hbar} I_{m-1,\hbar}(z_1, z_2, (z_1 + z_2 - z_1 z_2) e^{z\hbar^{1/2}})\,. \end{aligned} \tag{53}$$

Let us define $\widehat{J}_{m,\hbar}$ and $J_{m,\hbar}$ by the equation

$$\begin{aligned} \widehat{J}_{m,\hbar}(z_1, z_2) \;&=\; \frac{1}{\sqrt{(1 - z_1)(1 - z_2)}} \exp\Big(- \frac{\mathrm{Li}_2(z_1)}{\hbar} - \frac{\mathrm{Li}_2(z_2)}{\hbar}\Big) J_{m,\hbar}(z_1, z_2) \\ &=\; \Big\langle \frac{\sqrt{z_1 z_2 (1 - z_1)(1 - z_2)}}{z_1 + z_2 - z_1 z_2} \exp\Big(\frac{\delta}{2} z^2\Big) I_{m,\hbar}(z_1, z_2, z_0 e^{w\hbar^{1/2}}) \Big\rangle_{w,\delta}\,. \end{aligned} \tag{54}$$

If we multiply both sides of the Equations (53) with the factors from Equation (51), take the bracket of both sides, and apply Lemma 3.2 to change coordinates and the Gaussian, we find that $\widehat{J}_{\hbar,m}(z_1,z_2)$ satisfies the q–difference equations

$$\begin{aligned} \widehat{J}_{m,\hbar}(e^{\hbar}z_1,z_2) &= z_2^{-1}\widehat{J}_{m+1,\hbar}(z_1,z_2) - z_1 z_2^{-1}\widehat{J}_{m,\hbar}(z_1,z_2)\,, \\ \widehat{J}_{m,\hbar}(z_1,e^{\hbar}z_2) &= z_1^{-1}\widehat{J}_{m+1,\hbar}(z_1,z_2) - z_1^{-1} z_2\widehat{J}_{m,\hbar}(z_1,z_2)\,, \\ \widehat{J}_{m,\hbar}(z_1,z_2) &= z_1 z_2 e^{(m-1)\hbar}\widehat{J}_{m-2,\hbar}(z_1,z_2) - z_1 z_2 e^{(m-1)\hbar}\widehat{J}_{m-1,\hbar}(z_1,z_2)\,. \end{aligned} \tag{55}$$

From these relations we can derive the equations

$$\begin{aligned} 0 &= (z_1 - z_1 e^{(m+1)\hbar} + z_1^2 e^{(m+1)\hbar} - z_1)\widehat{J}_{m,\hbar}(z_1,z_2) \\ &\quad + (z_1 z_2 e^{(m+1)\hbar} - z_2 + q z_1)\widehat{J}_{m,\hbar}(e^{\hbar}z_1,z_2) + z_2\widehat{J}_{m,\hbar}(e^{2\hbar}z_1,z_2)\,, \\ 0 &= (z_2 - z_2 e^{(m+1)\hbar} + z_2^2 e^{(m+1)\hbar} - z_2)\widehat{J}_{m,\hbar}(z_1,z_2) \\ &\quad + (z_2 z_1 e^{(m+1)\hbar} - z_1 + q z_2)\widehat{J}_{m,\hbar}(z_1,e^{\hbar}z_2) + z_1\widehat{J}_{m,\hbar}(z_1,e^{2\hbar}z_2)\,. \end{aligned} \tag{56}$$

These q–difference equations give rise to equations for $J_{m,\hbar}$ given by

$$\begin{aligned} 0 &= (z_1 - z_1 e^{\hbar} + z_1^2 e^{\hbar} - z_1)J_{m,\hbar}(z_1,z_2) \\ &\quad + (z_1 z_2 e^{\hbar} - z_2 + q z_1)\sqrt{\frac{1-z_1}{1-e^{\hbar}z_1}}\exp\Big(\frac{\mathrm{Li}_2(z_1)-\mathrm{Li}_2(e^{\hbar}z_1)}{\hbar}\Big)J_{m,\hbar}(e^{\hbar}z_1,z_2) \\ &\quad + z_2\sqrt{\frac{1-z_1}{1-e^{2\hbar}z_1}}\exp\Big(\frac{\mathrm{Li}_2(z_1)-\mathrm{Li}_2(e^{2\hbar}z_1)}{\hbar}\Big)J_{m,\hbar}(e^{2\hbar}z_1,z_2)\,, \\ 0 &= (z_2 - z_2 e^{\hbar} + z_2^2 e^{\hbar} - z_2)J_{m,\hbar}(z_1,z_2) \\ &\quad + (z_2 z_1 e^{\hbar} - z_1 + q z_2)\sqrt{\frac{1-z_2}{1-e^{\hbar}z_2}}\exp\Big(\frac{\mathrm{Li}_2(z_2)-\mathrm{Li}_2(e^{\hbar}z_2)}{\hbar}\Big)J_{m,\hbar}(z_1,e^{\hbar}z_2) \\ &\quad + z_1\sqrt{\frac{1-z_2}{1-e^{2\hbar}z_2}}\exp\Big(\frac{\mathrm{Li}_2(z_2)-\mathrm{Li}_2(e^{2\hbar}z_2)}{\hbar}\Big)J_{m,\hbar}(z_1,e^{2\hbar}z_2)\,. \end{aligned} \tag{57}$$

These equations are also satisfied by the left hand side of Equation (49), which follows from repeated application of Equation (30).

It is easy to see that both sides of Equation (49) are formal power series in $\hbar$ with coefficients rational functions of (z_1,z_2) of nonpositive degree with respect to both z_1 and z_2, thus their coefficients embed in $\mathbb{Q}[\![z_1^{-1},z_2^{-1}]\!]$, and it suffices to prove (49) in the ring $\mathbb{Q}[\![z_1^{-1},z_2^{-1}]\!][\![\hbar]\!]$. In this case, we consider the linear q-difference equations (57) over the ring $\mathbb{Q}[\![z_1^{-1},z_2^{-1}]\!][\![\hbar]\!]$. Looking at the corresponding Newton polytopes, the leading monomials that appear as coefficients of this q–difference equation (57) at ∞ are $z_1^2(1+\mathrm{O}(z_1^{-1})\mathrm{O}(z_2^0))$ and $z_1^2 z_2(1+\mathrm{O}(z_1^{-1})\mathrm{O}(z_2^0))$ for the first equation, and $z_2^2(1+\mathrm{O}(z_1^0)\mathrm{O}(z_2^{-1}))$ and $z_1 z_2^2(1+\mathrm{O}(z_1^0)\mathrm{O}(z_2^{-1}))$ for the second equation, respectively. The structure of these monomials and their Newton polytopes shows that there is a unique solution to these equations in $\mathbb{Q}[\![z_1^{-1},z_2^{-1}]\!][\![\hbar]\!]$ determined by the value at $z_1=z_2=\infty$. Therefore, the identity in Equation (49) reduces to its specialization at $z_1=z_2=\infty$. Since

$$\begin{aligned} \psi_{\hbar}(0,\infty)\,\psi_{\hbar}(0,\infty) &= e^{\frac{\hbar}{6}} \quad \text{and} \\ e^{-\frac{\hbar}{24}}\big\langle \psi_{\hbar}(-w,0)^2\,\psi_{\hbar}(w,\infty)\big\rangle_{w,\delta} &= e^{-\frac{\hbar}{24}}\Big\langle \exp\Big(\frac{\hbar}{12}-\frac{z\hbar^{\frac12}}{2}\Big)\Big\rangle_{w,\delta} = e^{\frac{\hbar}{6}}\,, \end{aligned} \tag{58}$$

this completes the proof of Equation (49).

Going back to the general case of x, y, Equation (48) follows from Equation (34) together with (49) using Lemma 3.2 to shift the Gaussian and with a change of integration variable

$$w \mapsto w + \frac{1}{\hbar^{\frac{1}{2}}} \log\left(\frac{z_1 + z_2 - z_1 z_2}{z_1 e^{x_1 \hbar^{1/2}} + z_2 e^{x_2 \hbar^{1/2}} - z_1 z_2 e^{(x_1+x_2)\hbar^{1/2}}}\right) + x + y - \frac{x z_1 + y z_2}{z_0}\,. \tag{59}$$

The detailed calculation is given in Appendix B. □

3.5. **Inversion formula for $\psi_\hbar$.** In this section, we give an inversion formula for $\psi_\hbar$ analogous to the inversion formula (11) for the Faddeev quantum dilogarithm. Although we will not use this formula explicitly in our paper, we include it for completeness.

Lemma 3.7. We have:

$$\psi_\hbar(x,z)\frac{1 - z^{-1} e^{-x\hbar^{1/2}}}{1 - z^{-1}}\psi_\hbar(-x, 1/z) \;=\; \exp\Big(-\frac{x\hbar^{\frac{1}{2}}}{2} + \frac{\hbar}{12}\Big), \tag{60}$$

$$\widehat{\psi}_\hbar(0,z)\widehat{\psi}_\hbar(0,1/z) \;=\; \frac{\sqrt{-z}}{1-z}\exp\Big(\frac{\pi^2}{6\hbar} + \frac{1}{2\hbar}\log(-z)^2 + \frac{\hbar}{12}\Big)\,. \tag{61}$$

Proof. These formulas all follow from the well-known inversion formulas for the polylogarithms (see for example [30]):

$$\begin{aligned} \mathrm{Li}_2(z) + \mathrm{Li}_2(1/z) &= -\frac{\pi^2}{6} - \frac{1}{2}\log(-z)^2\,,\\ \mathrm{Li}_1(z) - \mathrm{Li}_1(1/z) &= -\log(-z)\,,\\ \mathrm{Li}_0(z) + \mathrm{Li}_0(1/z) &= -1\,,\\ \mathrm{Li}_{-n}(z) + (-1)^n\,\mathrm{Li}_{-n}(1/z) &= 0 \qquad (n>0)\,. \end{aligned} \tag{62}$$

Using these relations, we have

$$\begin{aligned} \widehat{\psi}_\hbar(0,z)\widehat{\psi}_\hbar(0,1/z) &= \exp\Big(-\sum_{k\in\mathbb{Z}_{\geq 0}} \frac{B_k\,\hbar^{k-1}}{k!}(\mathrm{Li}_{2-k}(z) + \mathrm{Li}_{2-k}(1/z))\Big)\\ &= \frac{\sqrt{-z}}{1-z}\exp\Big(\frac{\pi^2}{6\hbar} + \frac{1}{2\hbar}\log(-z)^2 + \frac{\hbar}{12}\Big)\,. \end{aligned} \tag{63}$$

Similarly, we find that

$$\begin{aligned} &\psi_\hbar(x,z)\frac{1 - z^{-1}e^{-x\hbar^{1/2}}}{1-z^{-1}}\psi_\hbar(-x,1/z)\\ &= \exp\Big(-\sum_{\substack{k,\ell\in\mathbb{Z}_{\geq 0}\\ k+\frac{\ell}{2}>1}} \frac{B_k\,x^\ell\,\hbar^{k+\frac{\ell}{2}-1}}{\ell!\,k!}\big(\mathrm{Li}_{2-k-\ell}(z) + (-1)^{k+\ell}\,\mathrm{Li}_{2-k-\ell}(1/z)\big)\Big)\\ &= \exp\Big(-\frac{x\hbar^{\frac{1}{2}}}{2} + \frac{\hbar}{12}\Big)\,. \end{aligned} \tag{64}$$

Notice that this is exactly the factor that appeared in the remark after Equation (1), which gave the relation between the $\psi_\hbar$ used in the current paper and the ones used in [11, Eq. 1.9]. Indeed, $\frac{1-z^{-1}}{1-z^{-1}e^{-x\hbar^{1/2}}}\psi_\hbar(-x,1/z)^{-1}$ is exactly the series used in [11] where the factor $\frac{1-z^{-1}}{1-z^{-1}e^{-x\hbar^{1/2}}}$ amounts to swapping the sign of B_1 as done there. $\square$

4. Elementary invariance properties

In this section, we review some basic choices needed to define the the Neumann–Zagier data for a triangulation with N tetrahedra, namely:

(a) an ordering of the N tetrahedra,
(b) an ordering of the N edges,
(c) an edge equation to remove,
(d) a path to represent the meridian curve,
(e) a flattening.

A change of these choices changes the corresponding Neumann–Zagier data in a simple way, which we now describe. Fix a triangulation and the choices needed to define Neumann–Zagier data

$$\Xi \;=\; (A,B,\nu,z,f,f'')\,. \tag{65}$$

Regarding choice (a), suppose that $\sigma\in S_N$ is a permutation (and also the associated matrix) of our labelling of the tetrahedra. Then the Neumann–Zagier matrices transform as follows

$$\Xi \;=\; (A,B,\nu,z,f,f'')\mapsto(A\sigma,B\sigma,\nu,\sigma^{-1}z,\sigma^{-1}f,\sigma^{-1}f'') \;=\; \Xi\cdot\sigma\,. \tag{66}$$

This implies that the integrand $f_\hbar^{\Xi}(x,z)$ of Equation (4) and the quadratic form Λ of Equation (6) satisfy

$$f_{\Xi\cdot\sigma,\hbar}(\sigma^{-1}x,\sigma^{-1}z) \;=\; f_\hbar^{\Xi}(x,z) \qquad \Lambda^{\Xi\cdot\sigma} \;=\; \sigma^t\Lambda^{\Xi}\sigma\,, \tag{67}$$

which combined with the fact that integration is invariant under a linear change of variables (see part (a) of Lemma 3.1), implies that $\Phi^{\Xi\cdot\sigma}(\hbar)=\Phi^{\Xi}(\hbar)$.

Choices (b), (c) and (d) are a special case of the following transformation of $P\in\mathrm{GL}_N(\mathbb{Z})$ acting on Neumann–Zagier data via

$$\Xi \;=\; (A,B,\nu,z,f,f'')\mapsto(PA,PB,P\nu,z,f,f'') \;=\; P\cdot\Xi\,. \tag{68}$$

It follows that the integrand and the quadratic form satisfy

$$f_\hbar^{P\cdot\Xi}(x,z) \;=\; f_\hbar^{\Xi}(x,z), \qquad \Lambda^{P\cdot\Xi} \;=\; \Lambda^{\Xi}\,, \tag{69}$$

which implies again that $\Phi^{\Xi\cdot\sigma}(\hbar)=\Phi^{\Xi}(\hbar)$.

Finally, if Ξ and $\widetilde{\Xi}$ differ by a choice of flattening, then it is easy to see that

$$f_\hbar^{\widetilde{\Xi}}(x,z) \;=\; e^{c\hbar}f_\hbar^{\Xi}(x,z), \qquad \Lambda^{\widetilde{\Xi}} \;=\; \Lambda^{\Xi} \tag{70}$$

for some $c\in\frac{1}{8}\mathbb{Z}$, which implies that $\Phi^{\widetilde{\Xi}}(\hbar)=e^{c\hbar}\,\Phi^{\Xi}(\hbar)$.

5. Invariance under the choice of quad

The definition of the series $\Phi^{\Xi}(\hbar)$ requires some choices, some of which were described and dealt with in the previous Section 4. What remains to complete the proof of Theorem 1.1 is the independence of $\Phi^{\Xi}(\hbar)$ under the choice of a nondegenerate quad, and the independence under the 2–3 Pachner moves that connect two ideal triangulations.

In this section, we will prove that $\Phi^{\Xi}(\hbar)$ is independent of the choice of a nondegenerate quad. Recall that to each pair of opposite edges of an ideal tetrahedron there is an associated variable called a shape variable. When defining Neumann–Zagier data (see Section 2.2), we must choose an edge for each tetrahedron, which associates one of these shapes. There are three possible choices, which leads to the action on $\mathbb{Z}/3\mathbb{Z}$ on the each column of the Neumann–Zagier data. All in all, for a triangulation with N tetrahedra, this leads to 3^N choices on of Neumann–Zagier data. On the other hand, the definition of the series $\Phi^{\Xi}(\hbar)$ requires a choice of a nondegenerate quad, i.e., one for which $\det(B) \neq 0$ (such quads always exist [11, Lem.A.3]). In this section, we will show that any of the 3^N choices of quad with $\det(B) \neq 0$ lead to the same $\Phi^{\Xi}(\hbar)$ series.

Theorem 5.1. *The series $\Phi^{\Xi}(\hbar)$ is independent of the choice of a nondegenerate quad.*

Proof. Fix two non–degenerate NZ data $\Xi = (A, B, \nu, z, f, f'')$ and $\widetilde{\Xi} = (\widetilde{A}, \widetilde{B}, \widetilde{\nu}, \widetilde{z}, \widetilde{f}, \widetilde{f}'')$ related by a quad change of the same ideal triangulation. The nondegeneracy assumption implies that $\det(B) \neq 0$ and $\det(\widetilde{B}) \neq 0$.

After reordering the tetrahedra (which does not change the $\Phi^{\Xi}(\hbar)$ series, as follows from Section 4), we can assume that $\widetilde{\Xi}$ is obtained by applying a change in quad that fixes the first N_0 shapes $z^{(0)}$, replaces the next N_1 shapes $z^{(1)}$ by $z'^{(1)}$ and replaces the next N_2 shapes $z^{(2)}$ by $z'^{(2)}$. (Recall that $z' = 1/(1-z)$ and $z'' = 1 - 1/z$).

This partitions the shapes $z = (z^{(0)}, z^{(1)}, z^{(2)})$ into three sets of size N_0, N_1, N_2 and the matrices A and B into three block matrices A_i and B_i of size $N \times N_i$ for $i = 0, 1, 2$

$$A \;=\; (A_0 \,|\, A_1 \,|\, A_2)\,, \qquad B \;=\; (B_0 \,|\, B_1 \,|\, B_2)\,, \tag{71}$$

and similarly for the flattening $f = (f^{(0)}, f^{(1)}, f^{(2)})$ and $f'' = (f''^{(0)}, f''^{(1)}, f''^{(2)})$. After the quad moves, the corresponding matrices and vectors are given by

$$\begin{aligned}
\tilde{A} \;&=\; (A_0 \,|\, -B_1 \,|\, -A_2 + B_2)\,, & \tilde{B} \;&=\; (B_0 \,|\, A_1 - B_1 \,|\, -A_2)\,, \\
\tilde{\nu} \;&=\; \nu - B_1 1 - A_2 1\,, & \tilde{z} \;&=\; (z^{(0)}, z'^{(1)}, z''^{(2)})\,, \\
\tilde{f} \;&=\; (f^{(0)}, 1 - f^{(1)} - f''^{(1)}, f''^{(2)})\,, & \tilde{f}'' \;&=\; (f''^{(0)}, f^{(2)}, 1 - f^{(2)} - f''^{(2)})\,.
\end{aligned} \tag{72}$$

We also partition the vector of formal Gaussian integration variables $x = (x^{(0)}, x^{(1)}, x^{(2)})$, as well as the symmetric matrix $Q := B^{-1}A$

$$Q \;=\; \begin{pmatrix} Q_{00} & Q_{01} & Q_{02} \\ Q_{01}^t & Q_{11} & Q_{12} \\ Q_{02}^t & Q_{12}^t & Q_{22} \end{pmatrix}\,. \tag{73}$$

With the above notation, we have

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_0 \rangle_{x, \Lambda_0} \tag{74}$$

where

$$I_0 \;=\; \exp\Big(\frac{\hbar}{8}f^tB^{-1}Af - \frac{\hbar^{\frac12}}{2}x^t(B^{-1}\nu-1)\Big)\prod_{j=1}^{N}\psi_\hbar(x_j,z_j)$$

and

$$\Lambda_0 \;=\; \mathrm{diag}(z') - Q\,.$$

Applying the first quad move as in Theorem 3.4 to the $\psi_\hbar$ with arguments of $z^{(1)}$ and the second quad move as in its Corollary 3.5 to the $\psi_\hbar$ with arguments $z^{(2)}$, we obtain that

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_1\rangle_{(x,y,w),\Lambda_1} \tag{75}$$

where

$$\begin{aligned} I_1 \;=\; \exp\Big(&\frac{\hbar}{8}f^tB^{-1}Af - \frac{\hbar^{\frac12}}{2}x^{(0)^t}(B^{-1}\nu-1)^{(0)} - \frac{N_1\hbar}{24} - \frac{\hbar^{\frac12}}{2}x^{(1)^t}(B^{-1}\nu-1)^{(1)} \\ &+\sum_{j=1}^{N_1}\Big(y_j+\frac{x_j^{(1)}z_j^{(1)}}{1-z_j^{(1)}}\Big)\frac{\hbar^{\frac12}}{2} + \frac{N_2\hbar}{24} - \frac{\hbar^{\frac12}}{2}x^{(2)^t}(B^{-1}\nu-1)^{(2)} - \sum_{j=1}^{N_2}x_j^{(2)}\frac{\hbar^{\frac12}}{2}\Big) \\ \times\prod_{j=1}^{N_0}\psi_\hbar(x_j^{(0)},z_j^{(0)})&\prod_{j=1}^{N_1}\psi_\hbar\Big(y_j+\frac{x_j^{(1)}z_j^{(1)}}{1-z_j^{(1)}},\frac{1}{1-z_j^{(1)}}\Big)\prod_{j=1}^{N_2}\psi_\hbar\Big(w_j-\frac{x_j^{(2)}}{1-z_j^{(2)}},1-z_j^{(2)^{-1}}\Big), \end{aligned}$$

y and w are vectors of size N_1 and N_2, respectively, and

$$\Lambda_1 \;=\; \begin{pmatrix}\Lambda & 0 & 0\\ 0 & \mathrm{diag}(z''^{(1)}) & 0\\ 0 & 0 & \mathrm{diag}(z^{(2)})-I\end{pmatrix}.$$

Making the change of variables $y_j \mapsto y_j - \frac{x_j^{(1)}z_j^{(1)}}{1-z_j^{(1)}}$ and $w_j \mapsto w_j + \frac{x_j^{(2)}}{1-z_j^{(2)}}$ and using Lemma 3.1 we obtain that

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_2\rangle_{(x,y,w),\Lambda_2} \tag{76}$$

where

$$\begin{aligned} I_2 \;=\; \exp\Big(&\frac{\hbar}{8}f^tB^{-1}Af - \frac{\hbar^{\frac12}}{2}x^{(0)^t}(B^{-1}\nu-1)^{(0)} - \frac{N_1\hbar}{24} - \frac{\hbar^{\frac12}}{2}x^{(1)^t}(B^{-1}\nu-1)^{(1)} \\ &+ y^t1\frac{\hbar^{\frac12}}{2} + \frac{N_2\hbar}{24} - \frac{\hbar^{\frac12}}{2}x^{(2)^t}(B^{-1}\nu)^{(2)}\Big) \\ &\times\prod_{j=1}^{N_0}\psi_\hbar(x_j^{(0)},z_j^{(0)})\prod_{j=1}^{N_1}\psi_\hbar\Big(y_j,\frac{1}{1-z_j^{(1)}}\Big)\prod_{j=1}^{N_2}\psi_\hbar\big(w_j,1-z_j^{(2)^{-1}}\big) \end{aligned}$$

and

$$\Lambda_2 \;=\; \left(\begin{array}{ccc|cc} \mathrm{diag}(z'^{(0)})-Q_{00} & -Q_{01} & -Q_{02} & 0 & 0\\ -Q_{01}^t & I-Q_{11} & -Q_{12} & I & 0\\ -Q_{02}^t & -Q_{12}^t & -Q_{22} & 0 & -I\\ \hline 0 & I & 0 & \mathrm{diag}(z''^{(1)}) & 0\\ 0 & 0 & -I & 0 & \mathrm{diag}(z^{(2)})-I \end{array}\right).$$

Note that $x^{(1)}$ and $x^{(2)}$ do not appear in the arguments of $\psi_\hbar$. Moreover, since $BQ = A$, we see that

$$B \begin{pmatrix} I & Q_{01} & Q_{02} \\ 0 & Q_{11}-I & Q_{12} \\ 0 & Q_{12}^t & Q_{22} \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{pmatrix} \;=\; (B_0 \,|\, A_1 - B_1 \,|\, -A_2) \;=\; \widetilde{B}\,, \tag{77}$$

which implies that

$$\det \begin{pmatrix} Q_{11}-I & Q_{12} \\ Q_{12}^t & Q_{22} \end{pmatrix} \neq 0\,, \quad \text{and} \quad \begin{pmatrix} Q_{11}-I & Q_{12} \\ Q_{12}^t & Q_{22} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & -I \end{pmatrix} \widetilde{B}^{-1} B \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix}. \tag{78}$$

Therefore, we can apply Fubini's Theorem (Lemma 3.1) with the integration variables $x^{(1)}, x^{(2)}$, and use Lemma 5.2 and the equality $Qf + f'' = B^{-1}\nu$ to obtain that

$$\Phi^{\Xi}(\hbar) \;=\; e^{c\hbar} \langle I_3 \rangle_{\widetilde{x},\Lambda_3}\,. \tag{79}$$

where $\widetilde{x} = (x^{(0)}, y, w)$, $c \in \frac{1}{24}\mathbb{Z}$

$$I_3 \;=\; \exp\Big(\frac{\hbar}{8} \widetilde{f}^t \widetilde{B}^{-1} \widetilde{A} \widetilde{f} - \frac{\hbar^{\frac{1}{2}}}{2} \widetilde{x}^t (\widetilde{B}^{-1} \widetilde{\nu} - 1)\Big) \prod_{j=1}^{N} \psi_\hbar(\widetilde{x}_j, \widetilde{z}_j)$$

and

$$\Lambda_3 \;=\; \operatorname{diag}(\widetilde{z}') - \widetilde{B}^{-1} \widetilde{A}\,.$$

The right hand side of Equation (79) is exactly equal to $e^{c\hbar}\Phi^{\widetilde{\Xi}}(\hbar)$, completing the proof of Theorem 5.1. ☐

Lemma 5.2. With the notation as in the proof of Theorem 5.1, we have the following identities

$$\widetilde{B}^{-1}\widetilde{A} \;=\; \begin{pmatrix} Q_{00} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{pmatrix} + \begin{pmatrix} Q_{01} & Q_{02} \\ -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} Q_{11}-I & Q_{12} \\ Q_{12}^t & Q_{22} \end{pmatrix}^{-1} \begin{pmatrix} Q_{01}^t & -I & 0 \\ Q_{02}^t & 0 & I \end{pmatrix}, \tag{80}$$

$$\begin{aligned} \widetilde{B}^{-1}\widetilde{\nu} - 1 \;=\; & \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (B^{-1}\nu - 1) - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ & + \begin{pmatrix} -Q_{01} & -Q_{02} \\ I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} Q_{11}-I & Q_{12} \\ Q_{12}^t & Q_{22} \end{pmatrix}^{-1} \left(\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} B^{-1}\nu - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \end{aligned} \tag{81}$$

and for some $d \in \mathbb{Z}$

$$\begin{aligned} \widetilde{f}\widetilde{B}^{-1}\widetilde{\nu} \;=\; & f^t B^{-1}\nu + d \\ & - \left((B^{-1}\nu)^t \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix} \right) \begin{pmatrix} Q_{11}-I & Q_{12} \\ Q_{12}^t & Q_{22} \end{pmatrix}^{-1} \left(\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} B^{-1}\nu - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right). \end{aligned} \tag{82}$$

Proof. We will show that boths sides of Equation (80) and Equation (81) are equal after multiplying by the invertible matrix $\widetilde{B}$. Denote

$$\begin{pmatrix} Q_{11}-I & Q_{12} \\ Q_{12}^t & Q_{22} \end{pmatrix}^{-1} \;=\; \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^t & \Gamma_{22} \end{pmatrix}, \tag{83}$$

and note that

$$\begin{aligned} &\widetilde{B}\begin{pmatrix} -Q_{01} & -Q_{02} \\ I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} Q_{11}-I & Q_{12} \\ Q_{12}^t & Q_{22} \end{pmatrix}^{-1} \;=\; \widetilde{B}\begin{pmatrix} -Q_{01}\Gamma_{11}-Q_{02}\Gamma_{12}^t & -Q_{01}\Gamma_{12}-Q_{02}\Gamma_{22} \\ \Gamma_{11} & \Gamma_{12} \\ -\Gamma_{12}^t & -\Gamma_{22} \end{pmatrix} \\ &= \big(-B_0Q_{01}\Gamma_{11}-B_0Q_{02}\Gamma_{12}^t+(A_1-B_1)\Gamma_{11}+A_2\Gamma_{12}^t \,\big|\, -B_0Q_{01}\Gamma_{12}-B_0Q_{02}\Gamma_{22}+(A_1-B_1)\Gamma_{12}+A_2\Gamma_{22}\big) \\ &= \big((B_1(Q_{11}-I)+B_2Q_{12}^t)\Gamma_{11}+(B_1Q_{12}+B_2Q_{22})\Gamma_{12}^t \,\big|\, (B_1(Q_{11}-I)+B_2Q_{12}^t)\Gamma_{12}+(B_1Q_{12}+B_2Q_{22})\Gamma_{22}\big) \\ &= (B_1 \mid B_2)\,. \end{aligned} \tag{84}$$

Therefore, since

$$\widetilde{B}\begin{pmatrix} Q_{00} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{pmatrix} \;=\; (B_0Q_{00} \mid 0 \mid -A_2)\,, \tag{85}$$

we see that multiplying the right hand side of Equations (80) on the left by $\widetilde{B}$ we have

$$(B_0Q_{00}+B_1Q_{01}^t+B_2Q_{02}^t \mid -B_1 \mid -A_2+B_2) \;=\; \widetilde{A}\,, \tag{86}$$

which completes the proof of Equation (80). Similarly, since

$$\widetilde{B}\begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}(B^{-1}\nu-1)-\widetilde{B}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \;=\; B_0(B^{-1}\nu)^{(0)}-B_01-A_11+B_11 \tag{87}$$

we see that multiplying the right hand side of Equations (81) on the left by $\widetilde{B}$ we have

$$B_0(B^{-1}\nu)^{(0)}-B_01-A_11+B_11+B_1(B^{-1}\nu)^{(1)}-B_11+B_2(B^{-1}\nu)^{(2)} \;=\; \widetilde{\nu}-\widetilde{B}1\,, \tag{88}$$

which completes the proof of Equation (81). For Equation (82), we will compute the three terms that appear there. Firstly, use Equation (81) to obtain that

$$\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & -I \end{pmatrix}(\widetilde{B}^{-1}\widetilde{\nu}-1) \;=\; -\begin{pmatrix} 1 \\ 0 \end{pmatrix}+\begin{pmatrix} Q_{11}-I & Q_{12} \\ Q_{12}^t & Q_{22} \end{pmatrix}^{-1}\left(\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}B^{-1}\nu-\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \tag{89}$$

to obtain that

$$\begin{aligned} &\left((B^{-1}\nu)^t\begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix}-\begin{pmatrix} 1 & 0 \end{pmatrix}\right)\left(\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & -I \end{pmatrix}(\widetilde{B}^{-1}\widetilde{\nu}-1)+\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\ &= ((B^{-1}\nu)^{(1)^t}-1)(\widetilde{B}^{-1}\widetilde{\nu})^{(1)}-(B^{-1}\nu)^{(2)^t}(\widetilde{B}^{-1}\widetilde{\nu})^{(2)}+(B^{-1}\nu)^{(2)^t}1\,. \end{aligned} \tag{90}$$

Secondly, expanding out

$$\begin{aligned} \widetilde{f}^t \widetilde{B}^{-1}\widetilde{\nu} &= f^{(0)}(\widetilde{B}^{-1}\widetilde{\nu})^{(0)} + (1 - f^{(1)} - f''^{(1)})(\widetilde{B}^{-1}\widetilde{\nu})^{(1)} + f''^{(2)}(\widetilde{B}^{-1}\widetilde{\nu})^{(2)} \\ &= f^{(0)}(\widetilde{B}^{-1}\widetilde{\nu})^{(0)} \\ &\quad + (1 - f^{(1)} - (B^{-1}\nu)^{(1)^t} + f^{(0)}Q_{01} + f^{(1)}Q_{11} + f^{(2)}Q_{12}^t)(\widetilde{B}^{-1}\widetilde{\nu})^{(1)} \\ &\quad + ((B^{-1}\nu)^{(2)^t} - f^{(0)}Q_{02} - f^{(1)}Q_{12} - f^{(2)}Q_{22})(\widetilde{B}^{-1}\widetilde{\nu})^{(2)} . \end{aligned} \tag{91}$$

Finally, using Equation (77) we obtain that

$$\begin{aligned} f^t B^{-1}\nu - f^{(1)}1 + f''^{(2)}1 &= f^t B^{-1}\nu - f^{(1)}1 - f^t Q(0,0,1)^t + (B^{-1}\nu)^{(2)^t}1 \\ &= f^t B^{-1}\widetilde{\nu} + (B^{-1}\nu)^{(2)^t}1 \\ &= f^{(0)}(\widetilde{B}^{-1}\widetilde{\nu})^{(0)} \\ &\quad + (f^{(0)}Q_{01} + f^{(1)}(Q_{11} - I) + f^{(2)}Q_{12}^t)(\widetilde{B}^{-1}\widetilde{\nu})^{(1)} \\ &\quad - (f^{(0)}Q_{02} + f^{(1)}Q_{12} + f^{(2)}Q_{22})(\widetilde{B}^{-1}\widetilde{\nu})^{(2)} + (B^{-1}\nu)^{(2)^t}1 . \end{aligned} \tag{92}$$

□

6. Invariance under Pachner moves

In this section, we will prove that Φ^{Ξ} is invariant under 2–3 Pachner moves. There are several versions of the 2–3 move (and of the corresponding pentagon identity in Teichmüller TQFT [27]) and the one we choose in the next theorem is slightly different from the one in [11, Sec.3.6] and can be related by composing with quad moves.

The 2–3 move involves two triangulations $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ with $N+2$ and $N+3$ tetrahedra, respectively, shown in Figure 2.

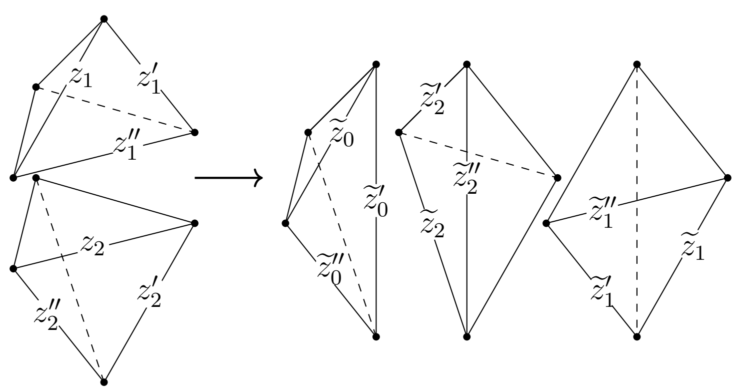

Figure 2. The 2–3 Pachner move.

Using $z_0 = z_1 + z_2 - z_1 z_2$ from Equation (46) used in Theorem 3.6, it follows that the shapes z of $\mathcal{T}$ and $\widetilde{z}$ of $\widetilde{\mathcal{T}}$ are related by

$$\begin{aligned} z = (z_1, z_2, z^*) \mapsto \widetilde{z} &= (\widetilde{z}_0,\ \widetilde{z}_1,\ \widetilde{z}_2,\ \widetilde{z}^*) \\ &= (z_0,\ z_1 z_0^{-1},\ z_2 z_0^{-1},\ z^*) . \end{aligned} \tag{93}$$

Similarly to [11, Eq. (3.20) and (3.21)], these shapes satisfy the relations

$$\begin{aligned} \widetilde{z}'_0 &= z'_1 z'_2\,, & \widetilde{z}''_1 &= z''_1 z_2\,, & \widetilde{z}''_2 &= z_1 z''_2\,, \\ z_1 &= \widetilde{z}_0 \widetilde{z}_1\,, & z'_1 &= \widetilde{z}'_1 \widetilde{z}_2\,, & z''_1 &= \widetilde{z}''_0 \widetilde{z}'_2\,, \\ z_2 &= \widetilde{z}_0 \widetilde{z}_2\,, & z'_2 &= \widetilde{z}_1 \widetilde{z}'_2\,, & z''_2 &= \widetilde{z}''_0 \widetilde{z}'_1\,. \end{aligned} \tag{94}$$

If we write the Neumann–Zagier matrices of $\mathcal{T}$ in the form

$$A \;=\; (a_1 \,|\, a_2 \,|\, a_*)\,, \qquad B =\; (b_1 \,|\, b_2 \,|\, b_*)\,, \tag{95}$$

where a_1, a_2 are the first two columns of A and a_* is the block $(N+2)\times N$ matrix of the remaining N columns of A, and likewise for B, then using Equations (94) the corresponding Neumann–Zagier matrices of $\widetilde{\mathcal{T}}$ are given by

$$\widetilde{A} \;=\; \begin{pmatrix} -1 & 0 & 0 & 0 \\ -b_1 - b_2 + a_1 + a_2 & a_1 - b_2 & a_2 - b_1 & a_* \end{pmatrix}, \qquad \widetilde{B} \;=\; \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & b_1 & b_2 & b_* \end{pmatrix} \tag{96}$$

and the corresponding vector $\widetilde{\nu} = (1, \nu)$. Analogously to the shapes, we will fix flattenings $(\widetilde{f}_0, \widetilde{f}_1, \widetilde{f}_2, \widetilde{f}^*)$ and $(\widetilde{f}''_0, \widetilde{f}''_1, \widetilde{f}''_2, \widetilde{f}''^*)$ for $(\widetilde{A} \,|\, \widetilde{B})$ and $(f_1, f_2, f^*) = (\widetilde{f}_0 + \widetilde{f}_1, \widetilde{f}_0 + \widetilde{f}_2, f^*)$ and $(f''_1, f''_2, f''^*) = (\widetilde{f}''_0 + \widetilde{f}'_2, \widetilde{f}''_0 + \widetilde{f}'_1, f''^*)$ for $(A \,|\, B)$. The data of the flattenings then satisfy the additive versions of Equations (94).

Theorem 6.1. *The series $\Phi^{\Xi}(\hbar)$ is invariant under 2–3 Pachner moves.*

The proof involves an application of the pentagon identity for $\psi_\hbar$ of Theorem 3.6.

For two triangulations $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ with $N+2$ and $N+3$ tetrahedra and NZ matrices $(\mathcal{A} \,|\, \mathcal{B})$, respectively, related by a Pachner 2–3 move. To define the corresponding series $\Phi^{\Xi}(\hbar)$ and $\Phi^{\widetilde{\Xi}}(\hbar)$, we need to possibly change quads on $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ so that both $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ are invertible. By a quad move q, we can replace $(\mathcal{A} \,|\, \mathcal{B})$ by $(\mathbf{A} \,|\, \mathbf{B})$ where $\det(\mathbf{B}) \neq 0$. Recalling that quad moves act on tetrahedra and actions on different tetrahedra commute, we can write the move $q = q_2 \times q_N$ as a product of quad moves q_2 on the first 2 tetrahedra times moves q_N on the remaining N tetrahedra of $\mathcal{T}$. Let $(A \,|\, B)$ denote the result of applying the move $1 \times q_N$ on the $N+2$ tetrahedra of $\mathcal{T}$.

Since the $(N+2)\times(N+2)$ matrix $\mathbf{B}$ has full rank and B and $\mathbf{B}$ have the same last N columns, it follows that B has nullity 0, 1 or 2.

Now on $\widetilde{\mathcal{T}}$, we can apply the identity move on the first three tetrahedra and the q_N moves on the remaining N tetrahedra, which transforms the NZ matrices $(\widetilde{\mathcal{A}} \,|\, \widetilde{\mathcal{B}})$ to $(\widetilde{A} \,|\, \widetilde{B})$, and further apply the q_2 moves on the second and third tetrahedra to obtain the NZ matrices $(\widetilde{\mathbf{A}} \,|\, \widetilde{\mathbf{B}})$. By looking at how the matrix B transforms under a 2–3 move (see Equation (96)), it follows that $\widetilde{\mathbf{B}}$ has the same rank as $\mathbf{B}$, and hence $\det(\widetilde{\mathbf{B}}) \neq 0$.

The above discussion can be summarized in the following commutative diagram.

$$
\begin{array}{ccccc}
(\mathcal{A}\mid\mathcal{B}) & & \xrightarrow{\quad 2\to 3 \quad} & & (\widetilde{\mathcal{A}}\mid\widetilde{\mathcal{B}}) \\
\Big\downarrow{\scriptstyle q_2\times q_N} & \searrow{\scriptstyle 1\times q_N} & & \swarrow{\scriptstyle 1\times 1\times q_N} & \Big\downarrow{\scriptstyle 1\times q_2\times q_N} \\
 & (A\mid B) & \xrightarrow{2\to 3} & (\widetilde{A}\mid\widetilde{B}) & \\
 & \swarrow{\scriptstyle q_2\times 1} & & \searrow{\scriptstyle 1\times q_2\times 1} & \\
(\mathbf{A}\mid\mathbf{B}) & & & & (\widetilde{\mathbf{A}}\mid\widetilde{\mathbf{B}})
\end{array} \tag{97}
$$

where $\det(\mathbf{B}) \neq 0$ and $\det(\widetilde{\mathbf{B}}) \neq 0$. Now $\Phi^{\Xi}(\hbar)$ and $\Phi^{\widetilde{\Xi}}(\hbar)$ can be defined using the non-degenerate NZ data with matrices $(\mathbf{A}\,|\,\mathbf{B})$ and $(\widetilde{\mathbf{A}}\,|\,\widetilde{\mathbf{B}})$, respectively. We will show that the two $\hbar$-series are equal using a diagram

$$\Phi^{\Xi}(\hbar) \xrightarrow{\otimes^i \text{Fourier}} \bullet \xrightarrow{\text{Pentagon}} \bullet \xrightarrow{\otimes^i \text{Fourier}^{-1}} \bullet \xrightarrow{2i-\text{Fubini}} \Phi^{\widetilde{\Xi}}(\hbar) \tag{98}$$

where $i = 0, 1, 2$ denotes the nullity of B. We will treat each case in a separate section. With this discussion the Theorem 6.1 follows from Propositions 6.2, 6.4, and 6.6. Before proceeding with the proof, we will set some notation. Similarly to the proof of the quad invariance, denote $\mathbf{Q} = \mathbf{B}^{-1}\mathbf{A}$

$$\mathbf{Q} \;=\; \left(\begin{array}{cc|c} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_1^* \\ \mathbf{Q}_{12} & \mathbf{Q}_{22} & \mathbf{Q}_2^* \\ \hline \mathbf{Q}_1^{*\,t} & \mathbf{Q}_2^{*\,t} & \mathbf{Q}^* \end{array}\right), \tag{99}$$

where $\mathbf{Q}_{ij}$ are matrices of size 1×1, $\mathbf{Q}_j^*$ are matrices of size $1 \times N$ and $\mathbf{Q}^*$ is a matrix of size $N \times N$.

6.1. **The case of B with full rank.** In this section, we prove Theorem 3.6 under the assumption that the matrix B has full rank. In this case, Equation (98) simplifies to the following one

$$\Phi^{\Xi}(\hbar) \xrightarrow{\text{Pentagon}} \Phi^{\widetilde{\Xi}}(\hbar) \tag{100}$$

since we do not need to apply Fourier transform, nor Fubini's theorem.

Proposition 6.2. If B has full rank, then $\Phi^{\Xi}(\hbar)$ is invariant under the 2–3 Pachner move given in Equation (96).

Proof. To begin with, noting that in this case $A = \mathbf{A}$ and $B = \mathbf{B}$, we have:

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_0 \rangle_{x,\Lambda} \tag{101}$$

where

$$I_0 \;=\; \exp\Big(\frac{\hbar}{8} f^t B^{-1} A f - \frac{\hbar^{\frac{1}{2}}}{2} x^t (B^{-1}\nu - 1)\Big) \prod_{j=1}^{N+2} \psi_\hbar(x_j, z_j)$$

and

$$\Lambda_0 \;=\; \operatorname{diag}(z') - \mathbf{Q}$$

and $x = (x_1, x_2, x^*)$ are the integration variables. Applying the pentagon identity of Theorem 3.6, to the $\psi_\hbar$ with arguments x_1, x_2 and introducing a new integration variable x_0, we obtain that

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_1 \rangle_{(x_0,x),\Lambda_1} \tag{102}$$

where

$$\begin{aligned} I_1 \;=\; & \exp\Big(-\frac{\hbar}{24} + \frac{\hbar}{8} f^t B^{-1} A f - \frac{\hbar^{\frac{1}{2}}}{2} x^t (B^{-1}\nu - 1) \Big)\, \psi_\hbar\Big(-x_0 - x_2 + \frac{x_1 z_2 + x_2 z_1}{z_0}, z_1 z_0^{-1} \Big) \\ & \times \psi_\hbar\Big(x_0 + x_1 + x_2 - \frac{x_1 z_2 + x_2 z_1}{z_0}, z_0 \Big)\, \psi_\hbar\Big(-x_0 - x_1 + \frac{x_1 z_2 + x_2 z_1}{z_0}, z_2 z_0^{-1} \Big) \prod_{j=1}^{N} \psi_\hbar(x_j^*, z_j^*) \end{aligned}$$

and

$$\Lambda_1 \;=\; \begin{pmatrix} \frac{(z_1+z_2-z_1z_2)^2}{(z_1-1)z_1(z_2-1)z_2} & 0 \\ 0 & \Lambda \end{pmatrix}.$$

Making a change of variables $x_0 \mapsto x_0 + x_1(-1 + z_2 z_0^{-1}) + x_2(-1 + z_1 z_0^{-1})$ using Lemma 3.1, we obtain that

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_2 \rangle_{(x_0,x),\Lambda_2}, \tag{103}$$

where

$$\begin{aligned} I_2 \;=\; & \exp\Big(-\frac{\hbar}{24} + \frac{\hbar}{8} f^t B^{-1} A f - \frac{\hbar^{\frac{1}{2}}}{2} x^t (B^{-1}\nu - 1) \Big)\, \psi_\hbar(-x_0 + x_1, z_1 z_0^{-1}) \\ & \times \psi_\hbar(x_0, z_0)\, \psi_\hbar(-x_0 + x_2, z_2 z_0^{-1}) \prod_{j=1}^{N} \psi_\hbar(x_j^*, z_j^*) \end{aligned}$$

and

$$\Lambda_2 \;=\; \left(\begin{array}{ccc|c} \frac{(z_1+z_2-z_1z_2)^2}{(z_1-1)z_1(z_2-1)z_2} & \frac{z_1+z_2-z_1z_2}{z_2(z_1-1)} & \frac{z_1+z_2-z_1z_2}{z_1(z_2-1)} & 0 \\ \frac{z_1+z_2-z_1z_2}{z_2(z_1-1)} & -\mathbf{Q}_{11} - \frac{z_1+z_2-z_1z_2}{z_2(z_1-1)} & 1 - \mathbf{Q}_{12} & -\mathbf{Q}_1^* \\ \frac{z_1+z_2-z_1z_2}{z_1(z_2-1)} & 1 - \mathbf{Q}_{12} & -\mathbf{Q}_{22} - \frac{z_1+z_2-z_1z_2}{z_1(z_2-1)} & -\mathbf{Q}_2^* \\ \hline 0 & -\mathbf{Q}_1^{*t} & -\mathbf{Q}_2^{*t} & \mathrm{diag}(z^{*\prime}) - \mathbf{Q}^* \end{array}\right).$$

Making a change of variables $x_1 \mapsto x_1 + x_0$ and $x_2 \mapsto x_2 + x_0$ using Lemma 3.1, and denoting $\widetilde{x} = (x_0, x)$, $\widetilde{z} = (z_0, z_1 z_0^{-1}, z_2 z_0^{-1}, z^*)$ we obtain that

$$\Phi^{\Xi}(\hbar) \;=\; e^{c\hbar} \langle I_3 \rangle_{\widetilde{x},\Lambda_3}, \tag{104}$$

where

$$I_3 \;=\; \exp\Big(\frac{\hbar}{8} \widetilde{f}^t \widetilde{B}^{-1} \widetilde{A} \widetilde{f} - \frac{\hbar^{\frac{1}{2}}}{2} \widetilde{x}^t (\widetilde{B}^{-1}\widetilde{\nu} - 1) \Big) \prod_{j=1}^{N+3} \psi_\hbar(\widetilde{x}_j, \widetilde{z}_j)$$

and

$$\Lambda_3 \;=\; \mathrm{diag}(\tilde{z}') - \tilde{B}^{-1}\tilde{A}.$$

□

The next lemma identifies vectors and matrices of the two triangulations $\mathcal{T}$ and $\widetilde{\mathcal{T}}$.

Lemma 6.3. With the notation used in the proof of the previous Proposition 6.2,

$$\widetilde{B}^{-1}\widetilde{A} = \left(\begin{array}{ccc|c} \mathbf{Q}_{11}+2\mathbf{Q}_{12}+\mathbf{Q}_{22}-1 & \mathbf{Q}_{11}+\mathbf{Q}_{12}-1 & \mathbf{Q}_{12}+\mathbf{Q}_{22}-1 & \mathbf{Q}^*_1+\mathbf{Q}^*_2 \\ \mathbf{Q}_{11}+\mathbf{Q}_{12}-1 & \mathbf{Q}_{11} & \mathbf{Q}_{12}-1 & \mathbf{Q}^*_1 \\ \mathbf{Q}_{12}+\mathbf{Q}_{22}-1 & \mathbf{Q}_{12}-1 & \mathbf{Q}_{22} & \mathbf{Q}^*_2 \\ \hline \mathbf{Q}^{*t}_1+\mathbf{Q}^{*t}_2 & \mathbf{Q}^{*t}_1 & \mathbf{Q}^{*t}_2 & \mathbf{Q}^* \end{array}\right) \tag{105}$$

$$\widetilde{B}^{-1}\widetilde{\nu} = \begin{pmatrix} (B^{-1}\nu)_1+(B^{-1}\nu)_2-1 \\ B^{-1}\nu \end{pmatrix} \tag{106}$$

$$\widetilde{f}^t\widetilde{B}^{-1}\widetilde{\nu} = f^tB^{-1}\nu-\widetilde{f}_0 \tag{107}$$

Proof. We will show that both sides of Equations (105) and Equations (106) are equal after multiplying by the invertible matrix $\widetilde{B}$. For Equation (105), we have

$$\begin{aligned} &\left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & b_1 & b_2 & b_* \end{array}\right) \left(\begin{array}{ccc|c} \mathbf{Q}_{11}+2\mathbf{Q}_{12}+\mathbf{Q}_{22}-1 & \mathbf{Q}_{11}+\mathbf{Q}_{12}-1 & \mathbf{Q}_{12}+\mathbf{Q}_{22}-1 & \mathbf{Q}^*_1+\mathbf{Q}^*_2 \\ \mathbf{Q}_{11}+\mathbf{Q}_{12}-1 & \mathbf{Q}_{11} & \mathbf{Q}_{12}-1 & \mathbf{Q}^*_1 \\ \mathbf{Q}_{12}+\mathbf{Q}_{22}-1 & \mathbf{Q}_{12}-1 & \mathbf{Q}_{22} & \mathbf{Q}^*_2 \\ \hline \mathbf{Q}^{*t}_1+\mathbf{Q}^{*t}_2 & \mathbf{Q}^{*t}_1 & \mathbf{Q}^{*t}_2 & \mathbf{Q}^* \end{array}\right) \\ &= \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ \substack{b_1(-1+\mathbf{Q}_{11}+\mathbf{Q}_{12})+b_*\mathbf{Q}^{*t}_1 \\ +b_2(-1+\mathbf{Q}_{12}+\mathbf{Q}_{22})+b_*\mathbf{Q}^{*t}_2} & \substack{b_1\mathbf{Q}_{11}+b_*\mathbf{Q}^{*t}_1 \\ +b_2(-1+\mathbf{Q}_{12})} & \substack{b_2\mathbf{Q}_{22}+b_*\mathbf{Q}^{*t}_2 \\ +b_1(-1+\mathbf{Q}_{12})} & b_1\mathbf{Q}^*_1+b_2\mathbf{Q}^*_2+b_*\mathbf{Q}_* \end{array}\right) \\ &= \widetilde{A}. \end{aligned} \tag{108}$$

For Equation (106), we have

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & b_1 & b_2 & b_* \end{pmatrix} \begin{pmatrix} (B^{-1}\nu)_1+(B^{-1}\nu)_2-1 \\ B^{-1}\nu \end{pmatrix} = \begin{pmatrix} 1 \\ \nu \end{pmatrix}. \tag{109}$$

Finally, for Equation (107),

$$\begin{aligned} \widetilde{f}^t\widetilde{B}^{-1}\nu &= \widetilde{f}^t \begin{pmatrix} (B^{-1}\nu)_1+(B^{-1}\nu)_2-1 \\ B^{-1}\nu \end{pmatrix} \\ &= (\widetilde{f}_0+\widetilde{f}_1)(B^{-1}\nu)_1+(\widetilde{f}_0+\widetilde{f}_2)(B^{-1}\nu)_2+f^*(B^{-1}\nu)_1-\widetilde{f}_0\,, \end{aligned} \tag{110}$$

and we recall we choose $f_i=\widetilde{f}_0+\widetilde{f}_i$ for $i=1,2$. □

6.2. **The case of B with nullity one.** In this section, we prove Theorem 3.6 under the assumption that the matrix B has nullity 1. In this case, starting from the series $\Phi^{\Xi}(\hbar)$, there are three intermediate formulas (shown as bullets) in Equation (98) that eventually identify the result with $\Phi^{\widetilde{\Xi}}(\hbar)$. The intermediate formulas involve a Fourier transform (adding one integration variable), a pentagon (adding a second variable), an inverse Fourier transform (adding a third), and an application of Fubini's theorem that removes two integration variables. The detailed computation is given in the next proposition.

Proposition 6.4. If B has rank $N+1$ then $\Phi^{\Xi}(\hbar)$ is invariant under the 2–3 Pachner move given in Equation (98).

Proof. Following the discussion above we can assume that $\mathrm{rank}(b_1 \,|\, b_2 \,|\, b_*) = \mathrm{rank}(b_2 \,|\, b_*) = N+1$. Then by [11, Lem.A.3], the matrix $(a_1 \,|\, b_2 \,|\, b_*)$ has full rank and we can apply a quad move to the first columns of $(A \,|\, B)$ to obtain

$$\mathbf{A} \;=\; (-b_1 \,|\, a_2 \,|\, a_*)\,, \qquad \mathbf{B} \;=\; (a_1 - b_1 \,|\, b_2 \,|\, b_*)\,, \qquad \boldsymbol{\nu} \;=\; \nu - b_1\,, \tag{111}$$

where $\mathbf{B}$ has full rank. The proof will use the following sequence of intermediate matrices

$$\begin{aligned}
&(\mathbf{A} \,|\, \mathbf{B} \,|\, \boldsymbol{\nu}) \;=\; \big(-b_1 \;\big|\; a_2 \;\big|\; a_* \;\big|\; a_1 - b_1 \;\big|\; b_2 \;\big|\; b_* \;\big|\; \nu - b_1\big) \\
&\quad\Big\downarrow\, q_1^{-1} \times 1 \\
&(A \,|\, B \,|\, \nu) \;=\; \big(a_1 \;\big|\; a_2 \;\big|\; a_* \;\big|\; b_1 \;\big|\; b_2 \;\big|\; b_* \;\big|\; \nu\big) \\
&\quad\Big\downarrow\, 2 \to 3 \\
&(\widetilde{A} \,|\, \widetilde{B} \,|\, \widetilde{\nu}) \;=\; \left(\begin{array}{c|c|c|c|c|c|c|c|c} -1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ a_1 + a_2 - b_1 - b_2 & a_1 - b_2 & a_2 - b_1 & a_* & 0 & b_1 & b_2 & b_* & \nu \end{array}\right) \\
&\quad\Big\downarrow\, 1 \times q_1 \times 1 \\
&(\widetilde{\mathbf{A}} \,|\, \widetilde{\mathbf{B}} \,|\, \widetilde{\boldsymbol{\nu}}) \;=\; \left(\begin{array}{c|c|c|c|c|c|c|c|c} -1 & -1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ a_1 + a_2 - b_1 - b_2 & -b_1 & a_2 - b_1 & a_* & 0 & a_1 - b_1 - b_2 & b_2 & b_* & \nu - b_1 \end{array}\right)
\end{aligned} \tag{112}$$

where $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ are invertible. With $x = (x_1, x_2, x^*)$ and $z = (z_1, z_2, z^*)$ vectors of size $N+2$, $\Phi^{\Xi}(\hbar)$ is defined by

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_0 \rangle_{(x_1, x_2, x^*), \mathbf{\Lambda}_0} \tag{113}$$

where with $\mathbf{f} = (f_1', f_2, f^*)$

$$I_0 \;=\; \exp\Big(\frac{\hbar}{8} \mathbf{f}^t \mathbf{B}^{-1} \mathbf{A} \mathbf{f} - \frac{\hbar^{\frac{1}{2}}}{2} x (\mathbf{B}^{-1} \boldsymbol{\nu} - 1)\Big)\, \psi_\hbar\Big(x_1, \frac{1}{1 - z_1}\Big)\, \psi_\hbar(x_2, z_2) \prod_{j=1}^{N} \psi_\hbar(x_j^*, z_j^*)$$

and

$$\mathbf{\Lambda}_0 \;=\; \mathrm{diag}(z_1'', z_2', z^{*\prime}) - \mathbf{Q}\,.$$

We apply Corollary 3.5 to the $\psi_\hbar$ with argument x_1 and introduce a new variable w_1 to obtain

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_1 \rangle_{(w_1, x_1, x_2, x^*), \mathbf{\Lambda}_1} \tag{114}$$

where

$$\begin{aligned}
I_1 \;=\; &\exp\Big(\frac{\hbar}{24} + \frac{\hbar}{8} \mathbf{f}^t \mathbf{B}^{-1} \mathbf{A} \mathbf{f} - \frac{\hbar^{\frac{1}{2}}}{2} x (\mathbf{B}^{-1} \boldsymbol{\nu} - 1) - x_1 \frac{\hbar^{\frac{1}{2}}}{2}\Big) \\
&\times \psi_\hbar\Big(w_1 - x_1(1 - z_1^{-1}), z_1\Big)\, \psi_\hbar(x_2, z_2) \prod_{j=1}^{N} \psi_\hbar(x_j^*, z_j^*)
\end{aligned}$$

and

$$\mathbf{\Lambda}_1 \;=\; \begin{pmatrix} \frac{z_1}{1-z_1} & 0 \\ 0 & \mathbf{\Lambda} \end{pmatrix}.$$

We substitute $w_1 \mapsto w_1 + x_1(1 - z_1^{-1})$ and obtain, from Lemma 3.1, that

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_2 \rangle_{(w_1,x_1,x_2,x^*),\mathbf{\Lambda}_2} \tag{115}$$

where

$$I_2 \;=\; \exp\Big(\frac{\hbar}{24} + \frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f} - \frac{\hbar^{\frac{1}{2}}}{2}x(\mathbf{B}^{-1}\boldsymbol{\nu} - 1) - x_1\frac{\hbar^{\frac{1}{2}}}{2}\Big)\, \psi_\hbar(w_1, z_1)\, \psi_\hbar(x_2, z_2) \prod_{j=1}^{N} \psi_\hbar(x_j^*, z_j^*)$$

and with $\mathbf{Q}_{11} = 0$ from Lemma 6.5

$$\mathbf{\Lambda}_2 \;=\; \left(\begin{array}{ccc|c} \frac{z_1}{1-z_1} & -1 & 0 & 0 \\ -1 & 0 & -\mathbf{Q}_{12} & -\mathbf{Q}_1^* \\ 0 & -\mathbf{Q}_{12} & \frac{1}{1-z_2} - \mathbf{Q}_{22} & -\mathbf{Q}_2^* \\ \hline 0 & -\mathbf{Q}_1^{*t} & -\mathbf{Q}_2^{*t} & z^{*\prime} - \mathbf{Q}^* \end{array}\right).$$

We apply the pentagon identity of Theorem 3.6 to the $\psi_\hbar$ with arguments w_1 and x_1 and obtain, with the new integration variable x_0, that

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_3 \rangle_{(x_0,w_1,x_1,x_2,x^*),\mathbf{\Lambda}_3} \tag{116}$$

where

$$I_3 \;=\; \exp\Big(\frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f} - \frac{\hbar^{\frac{1}{2}}}{2}x(\mathbf{B}^{-1}\boldsymbol{\nu} - 1) - x_1\frac{\hbar^{\frac{1}{2}}}{2}\Big)\, \psi_\hbar\Big(-x_0 - x_2 + \frac{w_1 z_2 + x_2 z_1}{z_0}, z_1 z_0^{-1}\Big)$$
$$\psi_\hbar\Big(x_0 + w_1 + x_2 - \frac{w_1 z_2 + x_2 z_1}{z_0}, z_0\Big)\, \psi_\hbar\Big(-x_0 - w_1 + \frac{w_1 z_2 + x_2 z_1}{z_0}, z_2 z_0^{-1}\Big) \prod_{j=1}^{N} \psi_\hbar(x_j^*, z_j^*)$$

and

$$\mathbf{\Lambda}_3 \;=\; \left(\begin{array}{cccc|c} \frac{(z_1+z_2-z_1z_2)^2}{(z_1-1)z_1(z_2-1)z_2} & 0 & 0 & 0 & 0 \\ 0 & \frac{z_1}{1-z_1} & -1 & 0 & 0 \\ 0 & -1 & 0 & -\mathbf{Q}_{12} & -\mathbf{Q}_1^* \\ 0 & 0 & -\mathbf{Q}_{12} & \frac{1}{1-z_2} - \mathbf{Q}_{22} & -\mathbf{Q}_2^* \\ \hline 0 & 0 & -\mathbf{Q}_1^{*t} & -\mathbf{Q}_2^{*t} & z^{*\prime} - \mathbf{Q}^* \end{array}\right).$$

We use Lemma 3.1 to change the variables $x_0 \mapsto x_0 - w_1 - x_2 + \frac{w_1 z_2 + x_2 z_1}{z_0}$ and obtain that

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_4 \rangle_{(x_0,w_1,x_1,x_2,x^*),\mathbf{\Lambda}_4} \tag{117}$$

where

$$I_4 \;=\; \exp\Big(\frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f} - \frac{\hbar^{\frac{1}{2}}}{2}x(\mathbf{B}^{-1}\boldsymbol{\nu} - 1) - x_1\frac{\hbar^{\frac{1}{2}}}{2}\Big)$$
$$\times\, \psi_\hbar(-x_0 + w_1, z_1 z_0^{-1})\, \psi_\hbar(x_0, z_0)\, \psi_\hbar(-x_0 + x_2, z_2 z_0^{-1}) \prod_{j=1}^{N} \psi_\hbar(x_j^*, z_j^*)$$

and

$$\mathbf{\Lambda}_4 \;=\; \left(\begin{array}{cccc|c} \frac{(z_1+z_2-z_1z_2)^2}{(z_1-1)z_1(z_2-1)z_2} & \frac{z_1+z_2-z_1z_2}{(z_1-1)z_2} & 0 & \frac{z_1+z_2-z_1z_2}{z_1(z_2-1)} & 0 \\ \frac{z_1+z_2-z_1z_2}{(z_1-1)z_2} & \frac{z_1}{z_2(1-z_1)} & -1 & 1 & 0 \\ 0 & -1 & 0 & -\mathbf{Q}_{12} & -\mathbf{Q}_1^* \\ \frac{z_1+z_2-z_1z_2}{z_1(z_2-1)} & 1 & -\mathbf{Q}_{12} & -\mathbf{Q}_{22}-\frac{z_1+z_2-z_1z_2}{z_1(z_2-1)} & -\mathbf{Q}_2^* \\ \hline 0 & 0 & -Q_1^{*t} & -\mathbf{Q}_2^{*t} & z^{*\prime}-\mathbf{Q}^* \end{array}\right).$$

We substitute $w_1 \mapsto w_1 + x_0$ and $x_2 \mapsto x_2 + x_0$ to obtain, with Lemma 3.1, that

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_5\rangle_{(x_0,w_1,x_1,x_2,x^*),\mathbf{\Lambda}_5} \tag{118}$$

where

$$\begin{aligned} I_5 \;=\; \exp\Big(\frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f} - \frac{\hbar^{\frac12}}{2}x(\mathbf{B}^{-1}\boldsymbol{\nu}-1) - \frac{\hbar^{\frac12}}{2}x_0(\mathbf{B}^{-1}\boldsymbol{\nu}_2-1) - x_1\frac{\hbar^{\frac12}}{2}\Big) \\ \times\, \psi_\hbar(w_1, z_1z_0^{-1})\,\psi_\hbar(x_0,z_0)\,\psi_\hbar(x_2,z_2z_0^{-1})\prod_{j=1}^N \psi_\hbar(x_j^*,z_j^*) \end{aligned}$$

and

$$\mathbf{\Lambda}_5 \;=\; \left(\begin{array}{cccc|c} -\mathbf{Q}_{22}+\frac{1}{(z_1-1)(z_2-1)} & 0 & -1-\mathbf{Q}_{12} & 1-\mathbf{Q}_{22} & -\mathbf{Q}_2^* \\ 0 & \frac{z_1}{z_2(1-z_1)} & -1 & 1 & 0 \\ -1-\mathbf{Q}_{12} & -1 & 0 & -\mathbf{Q}_{12} & -\mathbf{Q}_1^* \\ 1-\mathbf{Q}_{22} & 1 & -\mathbf{Q}_{12} & -\mathbf{Q}_{22}+\frac{z_1+z_2-z_1z_2}{z_1(z_2-1)} & -\mathbf{Q}_2^* \\ \hline -\mathbf{Q}_2^{*t} & 0 & -\mathbf{Q}_1^{*t} & -\mathbf{Q}_2^{*t} & z^{*\prime}-\mathbf{Q}^* \end{array}\right).$$

By applying the first quad move from Theorem 3.4 to $\psi_\hbar(w_1, z_1z_0^{-1})$ we obtain, with a new integration variable y_1, that

$$\Phi^{\Xi}(\hbar) \;=\; e^{-\frac{\hbar}{24}}\langle I_6\rangle_{(y_1,x_0,w_1,x_1,x_2,x^*),\mathbf{\Lambda}_6} \tag{119}$$

where

$$\begin{aligned} I_6 \;=\; \exp\Big(&-\frac{\hbar}{24} + \frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f} - \frac{\hbar^{\frac12}}{2}x(\mathbf{B}^{-1}\boldsymbol{\nu}-1) \\ &- \frac{\hbar^{\frac12}}{2}x_0((\mathbf{B}^{-1}\boldsymbol{\nu})_2-1) - x_1\frac{\hbar^{\frac12}}{2} + \Big(y_1 + \frac{w_1z_1z_0^{-1}}{1-z_1z_0^{-1}}\Big)\frac{\hbar^{\frac12}}{2}\Big) \\ &\times\, \psi_\hbar\Big(y_1 + \frac{w_1z_1z_0^{-1}}{1-z_1z_0^{-1}}, \frac{1}{1-z_1z_0^{-1}}\Big)\,\psi_\hbar(x_0,z_0)\,\psi_\hbar(x_2,z_2z_0^{-1})\prod_{j=1}^N \psi_\hbar(x_j^*,z_j^*) \end{aligned}$$

and

$$\mathbf{\Lambda}_6 = \left(\begin{array}{ccccc|c} \frac{(z_1-1)z_2}{z_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathbf{Q}_{22}+\frac{1}{(z_1-1)(z_2-1)} & 0 & -1-\mathbf{Q}_{12} & 1-\mathbf{Q}_{22} & -\mathbf{Q}_2^* \\ 0 & 0 & \frac{z_1}{z_2(1-z_1)} & -1 & 1 & 0 \\ 0 & -1-\mathbf{Q}_{12} & -1 & 0 & -\mathbf{Q}_{12} & -\mathbf{Q}_1^* \\ 0 & 1-\mathbf{Q}_{22} & 1 & -\mathbf{Q}_{12} & -\mathbf{Q}_{22}+\frac{z_1+z_2-z_1z_2}{z_1(z_2-1)} & -\mathbf{Q}_2^* \\ \hline 0 & -\mathbf{Q}_2^{*t} & 0 & -\mathbf{Q}_1^{*t} & -\mathbf{Q}_2^{*t} & z^{*\prime}-\mathbf{Q}^* \end{array}\right).$$

We change variables $y_1 \mapsto y_1 - w_1\frac{z_1z_0^{-1}}{1-z_1z_0^{-1}}$ using Lemma 3.1 and obtain that

$$\Phi^{\Xi}(\hbar) = \langle I_7\rangle_{(y_1,x_0,w_1,x_1,x_2,x^*),\mathbf{\Lambda}_7} \tag{120}$$

where

$$I_7 = \exp\Big(-\frac{\hbar}{24}+\frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f}-\frac{\hbar^{\frac12}}{2}x(\mathbf{B}^{-1}\boldsymbol{\nu}-1)-\frac{\hbar^{\frac12}}{2}x_0((\mathbf{B}^{-1}\boldsymbol{\nu})_2-1)-x_1\frac{\hbar^{\frac12}}{2}+y_1\frac{\hbar^{\frac12}}{2}\Big)$$
$$\times\,\psi_\hbar\Big(y_1,\frac{1}{1-z_1z_0^{-1}}\Big)\,\psi_\hbar(x_0,z_0)\,\psi_\hbar(x_2,z_2z_0^{-1})\prod_{j=1}^N\psi_\hbar(x_j^*,z_j^*)$$

and

$$\mathbf{\Lambda}_7 = \left(\begin{array}{ccccc|c} \frac{(z_1-1)z_2}{z_1} & 0 & 1 & 0 & 0 & 0 \\ 0 & -\mathbf{Q}_{22}+\frac{1}{(z_1-1)(z_2-1)} & 0 & -1-\mathbf{Q}_{12} & 1-\mathbf{Q}_{22} & -\mathbf{Q}_2^* \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1-\mathbf{Q}_{12} & -1 & 0 & -\mathbf{Q}_{12} & -\mathbf{Q}_1^* \\ 0 & 1-\mathbf{Q}_{22} & 1 & -\mathbf{Q}_{12} & -\mathbf{Q}_{22}+\frac{z_1+z_2-z_1z_2}{z_1(z_2-1)} & -\mathbf{Q}_2^* \\ \hline 0 & -\mathbf{Q}_2^{*t} & 0 & -\mathbf{Q}_1^{*t} & -\mathbf{Q}_2^{*t} & z^{*\prime}-\mathbf{Q}^* \end{array}\right).$$

Therefore, we can apply Fubini's Theorem (Lemma 3.1) with the integration variables w_1, x_1, to obtain that

$$\Phi^{\Xi}(\hbar) = \langle I_8\rangle_{(y_1,x_0,x_2,x^*),\mathbf{\Lambda}_8} \tag{121}$$

where

$$I_8 = \exp\Big(-\frac{\hbar}{24}+\frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f}-\frac{\hbar^{\frac12}}{2}(x_0((\mathbf{B}^{-1}\boldsymbol{\nu})_2-1)+y_1((\mathbf{B}^{-1}\boldsymbol{\nu})_1-1)$$
$$+x_2((\mathbf{B}^{-1}\boldsymbol{\nu})_2+(\mathbf{B}^{-1}\boldsymbol{\nu})_1-1)+x^*((\mathbf{B}^{-1}\boldsymbol{\nu})^*-1))\Big)$$
$$\times\,\psi_\hbar\Big(y_1,\frac{1}{1-z_1z_0^{-1}}\Big)\,\psi_\hbar(x_0,z_0)\,\psi_\hbar(x_2,z_2z_0^{-1})\prod_{j=1}^N\psi_\hbar(x_j^*,z_j^*)$$

and

$$\mathbf{\Lambda}_8 = \left(\begin{array}{ccc|c} \frac{(z_1-1)z_2}{z_1} & -\mathbf{Q}_{12}-1 & -\mathbf{Q}_{12} & -Q_1^* \\ -\mathbf{Q}_{12}-1 & \frac{1}{(z_1-1)(z_2-1)}-\mathbf{Q}_{22} & -\mathbf{Q}_{12}-\mathbf{Q}_{22} & -\mathbf{Q}_2^* \\ -\mathbf{Q}_{12} & -\mathbf{Q}_{12}-\mathbf{Q}_{22} & -2\mathbf{Q}_{12}-\mathbf{Q}_{22}+\frac{z_1+z_2-z_1z_2}{z_1(1-z_2)} & -\mathbf{Q}_1^*-\mathbf{Q}_2^* \\ \hline -\mathbf{Q}_1^{*t} & -\mathbf{Q}_2^{*t} & -\mathbf{Q}_1^{*t}-\mathbf{Q}_2^{*t} & z^{*\prime}-\mathbf{Q}^* \end{array}\right).$$

Using Lemma 6.5 we obtain, with respect to the integration variables $\tilde{x} = (x_0, y_1, x_2, x^*)$ and some $c \in \frac{1}{24}\mathbb{Z}$, that

$$\Phi^{\Xi}(\hbar) \;=\; e^{c\hbar} \langle I_9 \rangle_{\tilde{x}, \mathbf{\Lambda}_9} \tag{122}$$

where

$$I_9 \;=\; \exp\Big(\frac{\hbar}{8}\widetilde{\mathbf{f}}^t \widetilde{\mathbf{B}}^{-1}\widetilde{\mathbf{A}}\widetilde{\mathbf{f}} - \frac{\hbar^{\frac{1}{2}}}{2}\tilde{x}^t(\widetilde{\mathbf{B}}^{-1}\widetilde{\boldsymbol{\nu}} - 1)\Big) \prod_{j=0}^{N+2} \psi_\hbar(\widetilde{x}_j^*, \widetilde{z}_j^*)$$

and with $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{B}}$ as in (112)

$$\mathbf{\Lambda}_9 \;=\; \mathrm{diag}(\widetilde{z}_0', \widetilde{z}_1'', \widetilde{z}_2', \widetilde{z}^{*\prime}) - \widetilde{\mathbf{B}}^{-1}\widetilde{\mathbf{A}}.$$

□

Lemma 6.5. With the notation used in the previous proof of Proposition 6.4 we have $\mathbf{Q}_{11} = 0$ and the following equalities:

$$\widetilde{\mathbf{B}}^{-1}\widetilde{\mathbf{A}} \;=\; \left(\begin{array}{ccc|c} \mathbf{Q}_{22} & \mathbf{Q}_{12}+1 & \mathbf{Q}_{12}+\mathbf{Q}_{22} & \mathbf{Q}_2^* \\ \mathbf{Q}_{12}+1 & 0 & \mathbf{Q}_{12} & \mathbf{Q}_1^* \\ \mathbf{Q}_{12}+\mathbf{Q}_{22} & \mathbf{Q}_{12} & 2\mathbf{Q}_{12}+\mathbf{Q}_{22} & \mathbf{Q}_1^*+\mathbf{Q}_2^* \\ \hline \mathbf{Q}_2^{*t} & \mathbf{Q}_1^{*t} & \mathbf{Q}_1^{*t}+\mathbf{Q}_2^{*t} & \mathbf{Q}^* \end{array}\right), \tag{123}$$

$$\widetilde{\mathbf{B}}^{-1}\widetilde{\boldsymbol{\nu}} \;=\; \begin{pmatrix} (\mathbf{B}^{-1}\boldsymbol{\nu})_2 \\ (\mathbf{B}^{-1}\boldsymbol{\nu})_1 \\ (\mathbf{B}^{-1}\boldsymbol{\nu})_1 + (\mathbf{B}^{-1}\boldsymbol{\nu})_2 \\ (\mathbf{B}^{-1}\boldsymbol{\nu})^* \end{pmatrix}, \tag{124}$$

$$\widetilde{\mathbf{f}}^t\widetilde{\mathbf{B}}^{-1}\widetilde{\boldsymbol{\nu}} \;=\; \mathbf{f}^t\mathbf{B}^{-1}\boldsymbol{\nu}\,. \tag{125}$$

Proof. Similarly to the proof of Lemma 6.3 we will proof Equations (123) and (124) by showing that both sides are equal after multiplying by the invertible matrix $\widetilde{\mathbf{B}}$. Using $\mathbf{Q} = \mathbf{B}^{-1}\mathbf{A}$ and the fact that $(a_1 \,|\, b_2 \,|\, b_*)$ are linearly independent we conclude $\mathbf{Q}_{11} = 0$. For (123) we compute

$$\begin{aligned} &\begin{pmatrix} -1 & -1 & 1 & 0 \\ 0 & a_1 - b_1 - b_2 & b_2 & b_* \end{pmatrix} \left(\begin{array}{ccc|c} \mathbf{Q}_{22} & \mathbf{Q}_{12}+1 & \mathbf{Q}_{12}+\mathbf{Q}_{22} & \mathbf{Q}_2^* \\ \mathbf{Q}_{12}+1 & 0 & \mathbf{Q}_{12} & \mathbf{Q}_1^* \\ \mathbf{Q}_{12}+\mathbf{Q}_{22} & \mathbf{Q}_{12} & 2\mathbf{Q}_{12}+\mathbf{Q}_{22} & \mathbf{Q}_1^*+\mathbf{Q}_2^* \\ \hline \mathbf{Q}_2^{*t} & \mathbf{Q}_1^{*t} & \mathbf{Q}_1^{*t}+\mathbf{Q}_2^{*t} & \mathbf{Q}^* \end{array}\right) \\ &= \begin{pmatrix} -1 & -1 & 0 & 0 \\ \substack{(a_1-b_1)\mathbf{Q}_{12} \\ +b_2\mathbf{Q}_{22}+b_*\mathbf{Q}_2^{*t}} & b_2\mathbf{Q}_{12}+b_*\mathbf{Q}_1^{*t} & \substack{b_2\mathbf{Q}_{12}+b_*\mathbf{Q}_1^{*t}+ \\ (a_1-b_1)\mathbf{Q}_{12}+b_2\mathbf{Q}_{22}+b_*\mathbf{Q}_1^{*t}} & \substack{(a_1-b_1-b_2)\mathbf{Q}_1^* \\ +b_2(\mathbf{Q}_1^*+\mathbf{Q}_2^*)+b_*\mathbf{Q}^*} \end{pmatrix} \\ &= \widetilde{\mathbf{A}} \end{aligned} \tag{126}$$

For (124) we have

$$\begin{aligned}
&\begin{pmatrix} -1 & -1 & 1 & 0 \\ 0 & a_1 - b_1 - b_2 & b_2 & b_* \end{pmatrix} \begin{pmatrix} (\mathbf{B}^{-1}\boldsymbol{\nu})_2 \\ (\mathbf{B}^{-1}\boldsymbol{\nu})_1 \\ (\mathbf{B}^{-1}\boldsymbol{\nu})_1 + (\mathbf{B}^{-1}\boldsymbol{\nu})_2 \\ (\mathbf{B}^{-1}\boldsymbol{\nu})^* \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \boldsymbol{\nu} \end{pmatrix}
\end{aligned} \tag{127}$$

For Equation (125), we have

$$\begin{aligned}
&\widetilde{\mathbf{f}}^t \begin{pmatrix} (\mathbf{B}^{-1}\boldsymbol{\nu})_2 \\ (\mathbf{B}^{-1}\boldsymbol{\nu})_1 \\ (\mathbf{B}^{-1}\boldsymbol{\nu})_1 + (\mathbf{B}^{-1}\boldsymbol{\nu})_2 \\ (\mathbf{B}^{-1}\boldsymbol{\nu})^* \end{pmatrix} \\
&= (\widetilde{f}_1' + \widetilde{f}_2)(\mathbf{B}^{-1}\boldsymbol{\nu})_1 + (\widetilde{f}_0 - \widetilde{f}_2)(\mathbf{B}^{-1}\boldsymbol{\nu})_2 + \widetilde{f}^*(\mathbf{B}^{-1}\boldsymbol{\nu})^* \,.
\end{aligned} \tag{128}$$

Then, using the analogous relations between the flattening as given in Equation (94) for the shapes, we have $\widetilde{f}_1' + \widetilde{f}_2 = f_1' = \mathbf{f}_1$ and $\widetilde{f}_0 - \widetilde{f}_2 = f_2 = \mathbf{f}_2$. □

6.3. **The case of B with nullity two.** In this section, we prove Theorem 3.6 under the assumption that the matrix B has nullity 2. Similarly to the proof of Proposition 6.4, we use three intermediate formulas in Equation (98). They involve two Fourier transforms (adding two variables), a pentagon (adding a third variable), two inverse Fourier transforms (adding two variables) and an applications of Fubini's Theorem (removing four integration variables). The details are given in the following proposition.

Proposition 6.6. If B has rank N then $\Phi^{\Xi}(\hbar)$ is invariant under the 2–3 Pachner move given in Equation (98).

Proof. Following the discussion above, we can assume that $\mathrm{rank}(b_1 \,|\, b_2 \,|\, b_*) = \mathrm{rank}(\, b_*) = N$. Then by [11, Lem.A.3], the matrix $(a_1 - b_1 \,|\, a_2 - b_2 \,|\, b_*)$ has full rank and we can apply a quad move to the first columns of $(A \,|\, B)$ to obtain

$$\mathbf{A} \;=\; (-b_1 \,|\, -b_2 \,|\, a_*), \qquad \mathbf{B} \;=\; (a_1 - b_1 \,|\, a_2 - b_2 \,|\, b_*) \tag{129}$$

where $\mathbf{B}$ has full rank. The proof will use the following sequence of intermediate matrices

$$
\begin{aligned}
(\mathbf{A}\,|\,\mathbf{B}\,|\,\boldsymbol{\nu}) \;&=\; \big(-b_1 \,\big|\, -b_2 \,\big|\, a_* \,\big|\, a_1-b_1 \,\big|\, a_2-b_2 \,\big|\, b_* \,\big|\, \nu-b_1-b_2\big) \\
&\Big\downarrow\, q_2^{-1}\times 1 \\
(A\,|\,B\,|\,\nu) \;&=\; \big(a_1 \,\big|\, a_2 \,\big|\, a_* \,\big|\, b_1 \,\big|\, b_2 \,\big|\, b_* \,\big|\, \nu\big) \\
&\Big\downarrow\, 2\to 3 \\
(\widetilde{A}\,|\,\widetilde{B}\,|\,\widetilde{\nu}) \;&=\; \left(\begin{matrix}-1\\ a_1+a_2-b_1-b_2\end{matrix}\,\middle|\,\begin{matrix}0\\ a_1-b_2\end{matrix}\,\middle|\,\begin{matrix}0\\ a_2-b_1\end{matrix}\,\middle|\,\begin{matrix}0\\ a_*\end{matrix}\,\middle|\,\begin{matrix}-1\\ 0\end{matrix}\,\middle|\,\begin{matrix}1\\ b_1\end{matrix}\,\middle|\,\begin{matrix}1\\ b_2\end{matrix}\,\middle|\,\begin{matrix}0\\ b_*\end{matrix}\,\middle|\,\begin{matrix}1\\ \nu\end{matrix}\right) \\
&\Big\downarrow\, 1\times q_2\times 1 \\
(\widetilde{\mathbf{A}}\,|\,\widetilde{\mathbf{B}}\,|\,\widetilde{\boldsymbol{\nu}}) \;&= \\
&\left(\begin{matrix}-1\\ a_1+a_2-b_1-b_2\end{matrix}\,\middle|\,\begin{matrix}-1\\ -b_1\end{matrix}\,\middle|\,\begin{matrix}-1\\ -b_2\end{matrix}\,\middle|\,\begin{matrix}0\\ a_*\end{matrix}\,\middle|\,\begin{matrix}-1\\ 0\end{matrix}\,\middle|\,\begin{matrix}-1\\ a_1-b_1-b_2\end{matrix}\,\middle|\,\begin{matrix}-1\\ a_2-b_1-b_2\end{matrix}\,\middle|\,\begin{matrix}0\\ b_*\end{matrix}\,\middle|\,\begin{matrix}-1\\ \nu-b_1-b_2\end{matrix}\right)
\end{aligned}
\tag{130}
$$

where $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ are invertible. With $x=(x_1,x_2,x^*)$ and $z=(z_1,z_2,z^*)$ vectors of size $N+2$, $\Phi^{\Xi}(\hbar)$ is defined by

$$
\Phi^{\Xi}(\hbar) \;=\; \langle I_0\rangle_{(x_1,x_2,x^*),\boldsymbol{\Lambda}_0} \tag{131}
$$

where with $\mathbf{f}=(f_1',f_2',f^*)$

$$
\begin{aligned}
I_0 \;=\; &\exp\Big(\frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f}-\frac{\hbar^{\frac12}}{2}x^t(\mathbf{B}^{-1}\boldsymbol{\nu}-1)\Big)\\
&\times\psi_\hbar\Big(x_1,\frac{1}{1-z_1}\Big)\,\psi_\hbar\Big(x_2,\frac{1}{1-z_2}\Big)\,\prod_{j=1}^{N}\psi_\hbar(x_j^*,z_j^*)
\end{aligned}
$$

and

$$
\boldsymbol{\Lambda}_0 \;=\; \mathrm{diag}(1-z_1^{-1},1-z_2^{-1},z^{*\prime})-\mathbf{Q}\,.
$$

By applying Corollary 3.5 to both $\psi_\hbar(x_1,\frac{1}{1-z_1})$ and $\psi_\hbar(x_1,\frac{1}{1-z_1})$ we obtain with new integration variables w_1 and w_2 that

$$
\Phi^{\Xi}(\hbar) \;=\; \langle I_1\rangle_{(w_1,w_2,x_1,x_2,x^*),\boldsymbol{\Lambda}_1} \tag{132}
$$

where

$$
\begin{aligned}
I_1 \;=\; &\exp\Big(\frac{\hbar}{12}+\frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f}-\frac{\hbar^{\frac12}}{2}x^t(\mathbf{B}^{-1}\boldsymbol{\nu}-1)-(x_1+x_2)\frac{\hbar^{\frac12}}{2}\Big)\\
&\times\psi_\hbar\Big(w_1-x_1(1-z_1^{-1}),z_1\Big)\,\psi_\hbar\Big(w_2-x_2(1-z_2^{-1}),z_1\Big)\,\prod_{j=1}^{N}\psi_\hbar(x_j^*,z_j^*)
\end{aligned}
$$

and

$$\mathbf{\Lambda}_1 \;=\; \begin{pmatrix} \frac{z_2}{1-z_2} & 0 & 0 \\ 0 & \frac{z_1}{1-z_1} & 0 \\ 0 & 0 & \mathbf{\Lambda} \end{pmatrix}$$

The substitutions $w_1 \mapsto w_1 + x_1(1 - z_1^{-1})$ and $w_2 \mapsto w_2 + x_2(1 - z_2^{-1})$ imply according to Lemma 3.1 that

$$\Phi^\Xi(\hbar) \;=\; \langle I_2 \rangle_{(w_1,w_2,x_1,x_2,x^*),\mathbf{\Lambda}_2} \tag{133}$$

where

$$\begin{aligned} I_2 \;&=\; \exp\Big(\frac{\hbar}{12} + \frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f} - \frac{\hbar^{\frac12}}{2}x^t(\mathbf{B}^{-1}\boldsymbol{\nu} - 1) - (x_1+x_2)\frac{\hbar^{\frac12}}{2}\Big) \\ &\quad \times \psi_\hbar(w_1,z_1)\,\psi_\hbar(w_2,z_2)\prod_{j=1}^N \psi_\hbar(x_j^*,z_j^*) \end{aligned}$$

and with $\mathbf{Q}_{11} = \mathbf{Q}_{12} = \mathbf{Q}_{22} = 0$ from Lemma 6.7

$$\mathbf{\Lambda}_2 \;=\; \left(\begin{array}{cccc|c} \frac{z_1}{1-z_1} & 0 & -1 & 0 & 0 \\ 0 & \frac{z_2}{1-z_2} & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -\mathbf{Q}_1^* \\ 0 & -1 & 0 & 0 & -\mathbf{Q}_2^* \\ \hline 0 & 0 & -\mathbf{Q}_1^{*t} & -\mathbf{Q}_2^{*t} & z^{*\prime} - \mathbf{Q}^* \end{array}\right).$$

We apply the pentagon identity of Theorem 3.6 to the ψ_h with arguments w_1 and w_1 and obtain with the new integration variable x_0

$$\Phi^\Xi(\hbar) \;=\; \langle I_3 \rangle_{(x_0,w_1,w_2,x_1,x_2,x^*),\mathbf{\Lambda}_3} \tag{134}$$

where

$$\begin{aligned} I_3 \;&=\; \exp\Big(\frac{\hbar}{24} + \frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f} - \frac{\hbar^{\frac12}}{2}x^t(\mathbf{B}^{-1}\boldsymbol{\nu} - 1) - (x_1+x_2)\frac{\hbar^{\frac12}}{2}\Big) \\ &\quad \times \psi_\hbar\Big(-x_0 - w_2 + \frac{w_1z_2 + w_2z_1}{z_0}, z_1z_0^{-1}\Big)\,\psi_\hbar\Big(x_0 + w_1 + w_2 - \frac{w_1z_2+w_2z_1}{z_0}, z_0\Big) \\ &\quad \times \psi_\hbar\Big(-x_0 - w_1 + \frac{w_1z_2+w_2z_1}{z_0}, z_2z_0^{-1}\Big)\prod_{j=1}^N \psi_\hbar(x_j^*,z_j^*) \end{aligned}$$

and

$$\mathbf{\Lambda}_3 \;=\; \left(\begin{array}{ccccc|c} \frac{(z_1+z_2-z_1z_2)^2}{(z_1-1)z_1(z_2-1)z_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{z_1}{1-z_1} & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{z_2}{1-z_2} & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -\mathbf{Q}_1^* \\ 0 & 0 & -1 & 0 & 0 & -\mathbf{Q}_2^* \\ \hline 0 & 0 & 0 & -\mathbf{Q}_1^{*t} & -\mathbf{Q}_2^{*t} & z^{*\prime} - \mathbf{Q}^* \end{array}\right).$$

With change of variables $x_0 \mapsto x_0 - w_1 - w_2 + \frac{w_1z_2+w_2z_1}{z_0}$ Lemma 3.1 gives that

$$\Phi^\Xi(\hbar) \;=\; \langle I_4 \rangle_{(x_0,w_1,w_2,x_1,x_2,x^*),\mathbf{\Lambda}_4} \tag{135}$$

where

$$I_4 \;=\; \exp\Big(\frac{\hbar}{24}+\frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f}-\frac{\hbar^{\frac12}}{2}x^t(\mathbf{B}^{-1}\boldsymbol{\nu}-1)-(x_1+x_2)\frac{\hbar^{\frac12}}{2}\Big)$$
$$\times\,\psi_\hbar(-x_0+w_1,z_1z_0^{-1})\,\psi_\hbar(x_0,z_0)\,\psi_\hbar(-x_0+w_2,z_2z_0^{-1})\prod_{j=1}^{N}\psi_\hbar(x_j^*,z_j^*)$$

and

$$\boldsymbol{\Lambda}_4 \;=\; \left(\begin{array}{ccccc|c} \frac{(z_1+z_2-z_1z_2)^2}{(z_1-1)z_1(z_2-1)z_2} & \frac{z_1+z_2-z_1z_2}{(z_1-1)z_2} & \frac{z_1+z_2-z_1z_2}{(z_2-1)z_1} & 0 & 0 & 0\\ \frac{z_1+z_2-z_1z_2}{(z_1-1)z_2} & \frac{z_1}{1-z_1} & 1 & -1 & 0 & 0\\ \frac{z_1+z_2-z_1z_2}{(z_2-1)z_1} & 1 & \frac{z_2}{1-z_2} & 0 & -1 & 0\\ 0 & -1 & 0 & 0 & 0 & -\mathbf{Q}_1^*\\ 0 & 0 & -1 & 0 & 0 & -\mathbf{Q}_2^*\\ \hline 0 & 0 & 0 & -\mathbf{Q}_1^{*t} & -\mathbf{Q}_2^{*t} & z^{*\prime}-\mathbf{Q}^* \end{array}\right).$$

We substitute $w_1\mapsto w_1+x_0$ and $w_2\mapsto w_2+x_0$ to obtain with Lemma 3.1 that

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_5\rangle_{(x_0,w_1,w_2,x_1,x_2,x^*),\boldsymbol{\Lambda}_5} \tag{136}$$

where

$$I_5 \;=\; \exp\Big(\frac{\hbar}{24}+\frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f}-\frac{\hbar^{\frac12}}{2}x^t(\mathbf{B}^{-1}\boldsymbol{\nu}-1)-(x_1+x_2)\frac{\hbar^{\frac12}}{2}\Big)$$
$$\times\,\psi_\hbar(w_1,z_1z_0^{-1})\,\psi_\hbar(x_0,z_0)\,\psi_\hbar(w_2,z_2z_0^{-1})\prod_{j=1}^{N}\psi_\hbar(x_j^*,z_j^*)$$

and

$$\boldsymbol{\Lambda}_5 \;=\; \left(\begin{array}{ccccc|c} \frac{z_1+z_2-z_1z_2}{(z_1-1)(z_2-1)} & 0 & 0 & -1 & -1 & 0\\ 0 & \frac{z_1}{(1-z_1)z_2} & 1 & -1 & 0 & 0\\ 0 & 1 & \frac{z_2}{(1-z_2)z_1} & 0 & -1 & 0\\ -1 & -1 & 0 & 0 & 0 & -\mathbf{Q}_1^*\\ -1 & 0 & -1 & 0 & 0 & -\mathbf{Q}_2^*\\ \hline 0 & 0 & 0 & -\mathbf{Q}_1^{*t} & -\mathbf{Q}_2^{*t} & z^{*\prime}-\mathbf{Q}^* \end{array}\right).$$

By applying the first quad move from Theorem 3.4 to both $\psi_\hbar(w_1,z_1z_0^{-1})$ and $\psi_\hbar(w_2,z_2z_0^{-1})$ we obtain with new integration variables y_1 and y_2 that

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_6\rangle_{(y_1,y_2,x_0,w_1,w_2,x_1,x_2,x^*),\boldsymbol{\Lambda}_6} \tag{137}$$

where

$$I_6 \;=\; \exp\Big(-\frac{\hbar}{24}+\frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f}-\frac{\hbar^{\frac12}}{2}x^t(\mathbf{B}^{-1}\boldsymbol{\nu}-1)-(x_1+x_2)\frac{\hbar^{\frac12}}{2}$$
$$+\Big(y_1+\frac{w_1z_1z_0^{-1}}{1-z_1z_0^{-1}}\Big)\frac{\hbar^{\frac12}}{2}+\Big(y_2+\frac{w_2z_2z_0^{-1}}{1-z_2z_0^{-1}}\Big)\frac{\hbar^{\frac12}}{2}\Big)\prod_{j=1}^{N}\psi_\hbar(x_j^*,z_j^*)$$
$$\times\,\psi_\hbar\Big(y_1+\frac{w_1z_1z_0^{-1}}{1-z_1z_0^{-1}},\frac{1}{1-z_1z_0^{-1}}\Big)\,\psi_\hbar\Big(y_2+\frac{w_2z_2z_0^{-1}}{1-z_2z_0^{-1}},\frac{1}{1-z_2z_0^{-1}}\Big)\,\psi_\hbar(x_0,z_0)$$

and

$$\mathbf{\Lambda}_6 \;=\; \left(\begin{array}{ccccccc|c} \frac{(z_1-1)z_2}{z_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{(z_2-1)z_1}{z_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{z_1+z_2-z_1z_2}{(z_1-1)(z_2-1)} & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & \frac{z_1}{(1-z_1)z_2} & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{z_2}{(1-z_2)z_1} & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & -\mathbf{Q}_1^* \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & -\mathbf{Q}_2^* \\ \hline 0 & 0 & 0 & 0 & 0 & -\mathbf{Q}_1^{*t} & -\mathbf{Q}_2^{*t} & z^{*\prime} - \mathbf{Q}^* \end{array}\right).$$

We change variables $y_1 \mapsto y_1 - w_1 \frac{z_1 z_0^{-1}}{1-z_1 z_0^{-1}}$ and $y_2 \mapsto y_2 - w_2 \frac{z_2 z_0^{-1}}{1-z_2 z_0^{-1}}$ using Lemma 3.1 to obtain that

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_7 \rangle_{(y_1,y_2,x_0,w_1,w_2,x_1,x_2,x^*),\mathbf{\Lambda}_7} \tag{138}$$

where

$$\begin{aligned} I_7 \;=\; & \exp\Big(-\frac{\hbar}{24} + \frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f} - \frac{\hbar^{\frac{1}{2}}}{2} x^t(\mathbf{B}^{-1}\boldsymbol{\nu} - 1) - (x_1+x_2)\frac{\hbar^{\frac{1}{2}}}{2} + (y_1+y_2)\frac{\hbar^{\frac{1}{2}}}{2} \Big) \\ & \times \psi_\hbar\Big(y_1, \frac{1}{1-z_1 z_0^{-1}}\Big)\, \psi_\hbar\Big(y_2, \frac{1}{1-z_2 z_0^{-1}}\Big)\, \psi_\hbar(x_0, z_0) \prod_{j=1}^{N} \psi_\hbar(x_j^*, z_j^*) \end{aligned}$$

and

$$\mathbf{\Lambda}_7 \;=\; \left(\begin{array}{ccccccc|c} \frac{(z_1-1)z_2}{z_1} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{(z_2-1)z_1}{z_2} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{z_1+z_2-z_1z_2}{(z_1-1)(z_2-1)} & 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & -\mathbf{Q}_1^* \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & -\mathbf{Q}_2^* \\ \hline 0 & 0 & 0 & 0 & 0 & -\mathbf{Q}_1^{*t} & -\mathbf{Q}_2^{*t} & z^{*\prime} - \mathbf{Q}^* \end{array}\right).$$

Therefore, we can apply Fubini's Theorem (Lemma 3.1) with the integration variables w_1, w_2, x_1, x_2, to obtain that

$$\Phi^{\Xi}(\hbar) \;=\; \langle I_8 \rangle_{(y_1,y_2,x_0,x^*),\mathbf{\Lambda}_8} \tag{139}$$

where

$$\begin{aligned} I_8 \;=\; \exp\Big(& -\frac{\hbar}{4}(\mathbf{B}^{-1}\boldsymbol{\nu})_1(\mathbf{B}^{-1}\boldsymbol{\nu})_2 - \frac{\hbar}{24} + \frac{\hbar}{8}\mathbf{f}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{f} + \frac{\hbar^{\frac{1}{2}}}{2}\big(x_0\left((\mathbf{B}^{-1}\boldsymbol{\nu})_1 + (\mathbf{B}^{-1}\boldsymbol{\nu})_2\right)\big) \\ & - \frac{\hbar^{\frac{1}{2}}}{2}y_1\left((\mathbf{B}^{-1}\boldsymbol{\nu})_1 - 1\right) - \frac{\hbar^{\frac{1}{2}}}{2}y_2\left((\mathbf{B}^{-1}\boldsymbol{\nu})_2 - 1\right) \\ & - \frac{\hbar^{\frac{1}{2}}}{2}x^{*t}\left((\mathbf{B}^{-1}\boldsymbol{\nu})^* - (\mathbf{B}^{-1}\boldsymbol{\nu})_2\mathbf{Q}_1^* - (\mathbf{B}^{-1}\boldsymbol{\nu})_1\mathbf{Q}_2^* - 1\right)\Big) \\ \times\, \psi_\hbar\Big(y_1, & \frac{1}{1-z_1z_0^{-1}}\Big)\,\psi_\hbar\Big(y_2,\frac{1}{1-z_2z_0^{-1}}\Big)\,\psi_\hbar(x_0,z_0)\prod_{j=1}^{N}\psi_\hbar(x_j^*,z_j^*) \end{aligned}$$

and

$$\boldsymbol{\Lambda}_8 \;=\; \left(\begin{array}{ccc|c} \frac{(z_1-1)z_2}{z_1} & 0 & -1 & -\mathbf{Q}_1^* \\ 0 & \frac{(z_2-1)z_1}{z_2} & -1 & -\mathbf{Q}_2^* \\ -1 & -1 & \frac{z_1+z_2-z_2z_2}{(z_1-1)(z_2-1)}+2 & \mathbf{Q}_1^*+\mathbf{Q}_2^* \\ \hline -\mathbf{Q}_1^{*t} & -\mathbf{Q}_2^{*t} & \mathbf{Q}_1^{*t}+\mathbf{Q}_2^{*t} & -\mathbf{Q}^*+\mathbf{Q}_1^*\mathbf{Q}_2^*+\mathbf{Q}_2^*\mathbf{Q}_1^* \end{array}\right).$$

Using Lemma 6.7, we obtain with respect to the integration variables $\tilde{x} = (x_0, y_1, y_2, x^*)$ and some $c \in \frac{1}{24}\mathbb{Z}$ that

$$\Phi^{\Xi}(\hbar) \;=\; e^{c\hbar}\langle I_9\rangle_{\tilde{x},\boldsymbol{\Lambda}_9} \tag{140}$$

where with $\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}$ and $\widetilde{\boldsymbol{\nu}}$ as in Equation (130) and

$$I_9 \;=\; \exp\Big(\frac{\hbar}{8}\widetilde{\mathbf{f}}^t\widetilde{\mathbf{B}}^{-1}\widetilde{\mathbf{A}}\widetilde{\mathbf{f}} - \frac{\hbar^{\frac{1}{2}}}{2}\tilde{x}^t(\widetilde{\mathbf{B}}^{-1}\widetilde{\boldsymbol{\nu}} - 1)\Big)\prod_{j=0}^{N+2}\psi_\hbar(\widetilde{x}_j^*, \widetilde{z}_j^*) \tag{141}$$

and

$$\boldsymbol{\Lambda}_9 \;=\; \mathrm{diag}(\widetilde{z}_0',\, \widetilde{z}_1'',\, \widetilde{z}_2'',\, \widetilde{z}^{*\prime}) - \widetilde{\mathbf{B}}^{-1}\widetilde{\mathbf{A}}. \tag{142}$$

□

Lemma 6.7. With the notation used in the proof of the previous Proposition 6.6 we have $\mathbf{Q}_{11} = \mathbf{Q}_{12} = \mathbf{Q}_{22} = 0$ and the following equalities:

$$\widetilde{\mathbf{B}}^{-1}\widetilde{\mathbf{A}} \;=\; \left(\begin{array}{ccc|c} -1 & 1 & 1 & -\mathbf{Q}_1^*-\mathbf{Q}_2^* \\ 1 & 0 & 0 & \mathbf{Q}_1^* \\ 1 & 0 & 0 & \mathbf{Q}_2^* \\ \hline -\mathbf{Q}_1^{*t}-\mathbf{Q}_2^{*t} & \mathbf{Q}_1^{*t} & \mathbf{Q}_2^{*t} & \mathbf{Q}^*-\mathbf{Q}_1^{*t}\mathbf{Q}_2^*-\mathbf{Q}_2^{*t}\mathbf{Q}_1^* \end{array}\right), \tag{143}$$

$$\widetilde{\mathbf{B}}^{-1}\widetilde{\boldsymbol{\nu}} \;=\; \begin{pmatrix} -(\mathbf{B}^{-1}\boldsymbol{\nu})_1 - (\mathbf{B}^{-1}\boldsymbol{\nu})_2 + 1 \\ (\mathbf{B}^{-1}\boldsymbol{\nu})_1 \\ (\mathbf{B}^{-1}\boldsymbol{\nu})_2 \\ (\mathbf{B}^{-1}\boldsymbol{\nu})^* - (\mathbf{B}^{-1}\boldsymbol{\nu})_1\mathbf{Q}_2^* - (\mathbf{B}^{-1}\boldsymbol{\nu})_2\mathbf{Q}_1^* \end{pmatrix}, \tag{144}$$

$$\widetilde{\mathbf{f}}^t\widetilde{\mathbf{B}}^{-1}\widetilde{\boldsymbol{\nu}} \;=\; \mathbf{f}^t\mathbf{B}^{-1}\boldsymbol{\nu} + \widetilde{f}_0 - 2(\mathbf{B}^{-1}\boldsymbol{\nu})_1(\mathbf{B}^{-1}\boldsymbol{\nu})_2\,. \tag{145}$$

Proof. The relation $\mathbf{B}^{-1}\mathbf{A} = \mathbf{Q}$ and the fact that the columns of $(a_1 \,|\, a_2 \,|\, b_*)$ are linearly independent imply that $\mathbf{Q}_{11} = \mathbf{Q}_{12} = \mathbf{Q}_{22} = 0$. We will prove Equation (143) by showing the identity after multiplying by the invertible matrix $\widetilde{\mathbf{B}}$. For (143) we compute

$$\begin{aligned}
&\left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & a_1-b_1-b_2 & a_2-b_1-b_2 & b_* \end{array}\right)\left(\begin{array}{ccc|c} -1 & 1 & 1 & -\mathbf{Q}_1^* - \mathbf{Q}_2^* \\ 1 & 0 & 0 & \mathbf{Q}_1^* \\ 1 & 0 & 0 & \mathbf{Q}_2^* \\ \hline -\mathbf{Q}_1^{*t} - \mathbf{Q}_2^{*t} & \mathbf{Q}_1^{*t} & \mathbf{Q}_2^{*t} & \mathbf{Q}^* - \mathbf{Q}_1^{*t}\mathbf{Q}_2^* - \mathbf{Q}_2^{*t}\mathbf{Q}_1^* \end{array}\right) \\
&= \left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ \hline \begin{smallmatrix} a_1+a_2-2b_1-2b_2 \\ -b_*\mathbf{Q}_1^{*t}-b_*\mathbf{Q}_2^{*t} \end{smallmatrix} & b_*\mathbf{Q}_1^{*t} & b_*\mathbf{Q}_2^{*t} & \begin{smallmatrix} (a_1-b_1)\mathbf{Q}_1^{*t}+(a_2-b_2)\mathbf{Q}_2^{*t} \\ -b_2\mathbf{Q}_1^{*t}-b_1\mathbf{Q}_2^{*t}+b_*\mathbf{Q}^*-b_*\mathbf{Q}_1^{*t}\mathbf{Q}_2^*-b_*\mathbf{Q}_2^{*t}\mathbf{Q}_1^* \end{smallmatrix} \end{array}\right) \\
&= \widetilde{\mathbf{A}}.
\end{aligned} \tag{146}$$

For Equation (144), we compute

$$\begin{aligned}
&\left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & a_1-b_1-b_2 & a_2-b_1-b_2 & b_* \end{array}\right)\begin{pmatrix} -(\mathbf{B}^{-1}\boldsymbol{\nu})_1 - (\mathbf{B}^{-1}\boldsymbol{\nu})_2 + 1 \\ (\mathbf{B}^{-1}\boldsymbol{\nu})_1 \\ (\mathbf{B}^{-1}\boldsymbol{\nu})_2 \\ (\mathbf{B}^{-1}\boldsymbol{\nu})^* - (\mathbf{B}^{-1}\boldsymbol{\nu})_1\mathbf{Q}_2^* - (\mathbf{B}^{-1}\boldsymbol{\nu})_2\mathbf{Q}_1^* \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ \boldsymbol{\nu} \end{pmatrix}.
\end{aligned} \tag{147}$$

For Equation (145), we note that for $i = 1, 2$ we have

$$\widetilde{\mathbf{f}}^{*t}\mathbf{Q}_i^* \;=\; (\widetilde{\mathbf{B}}^{-1}\widetilde{\boldsymbol{\nu}})_i - \widetilde{\mathbf{f}}_0 - \widetilde{\mathbf{f}}_i'' \tag{148}$$

and so

$$\widetilde{\mathbf{f}}^t\widetilde{\mathbf{B}}^{-1}\widetilde{\boldsymbol{\nu}} \;=\; \widetilde{f}_0 + (\widetilde{f}_1' + \widetilde{f}_2)(\mathbf{B}^{-1}\boldsymbol{\nu})_1 + (\widetilde{f}_1 + \widetilde{f}_2')(\mathbf{B}^{-1}\boldsymbol{\nu})_2 + \widetilde{f}^{*t}(\mathbf{B}^{-1}\boldsymbol{\nu})^* - 2(\mathbf{B}^{-1}\boldsymbol{\nu})_1(\mathbf{B}^{-1}\boldsymbol{\nu})_2 \tag{149}$$

Then, using the analogous relations between the flattening as given in Equation (94) for the shapes, we have $\widetilde{f}_1' + \widetilde{f}_2 = f_1' = \mathbf{f}_1$ and $\widetilde{f}_1 + \widetilde{f}_2' = f_2' = \mathbf{f}_2$. ☐

7. The series of the simplest hyperbolic 4_1 knot

In this section, we discuss an effective computation of the power series $\Phi^{\Xi}(\hbar)$ for the simplest hyperbolic knot, namely the 4_1 knot. This example was studied extensively in [11]. From [11, Ex. 2.6], we obtain that the NZ datum Ξ_{4_1} is given by

$$A \;=\; \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \quad B \;=\; \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \nu \;=\; \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad f \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f'' \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{150}$$

and $z_1 = z_2 = \zeta_6 = e^{2\pi i/6}$. Therefore, we have

$$\Phi^{\Xi_{4_1}}(\hbar) \;=\; e^{\frac{\hbar}{8}}\langle \psi_\hbar(x_1, \zeta_6)\psi_\hbar(x_2, \zeta_6)\rangle_{(x_1,x_2),\Lambda_0}\,, \tag{151}$$

where

$$\Lambda_0 \;=\; \begin{pmatrix} \zeta_6 - 1 & -1 \\ -1 & \zeta_6 - 1 \end{pmatrix}. \tag{152}$$

This is a two dimensional formal Gaussian integral which can be simplified. Using $\zeta_6 = 1/(1-\zeta_6) = 1-\zeta_6^{-1}$ and applying a change of coordinates $x_1 \mapsto x_1 - \zeta_6 x_2$, we find that

$$\Phi^{\Xi_{4_1}}(\hbar) \;=\; e^{\frac{\hbar}{8}} \langle \psi_\hbar(x_1-\zeta_6 x_2,\zeta_6)\psi_\hbar(x_2,\zeta_6)\rangle_{(x_1,x_2),\Lambda_1}\,, \tag{153}$$

where

$$\Lambda_1 \;=\; \begin{pmatrix} \zeta_6-1 & 0 \\ 0 & 2\zeta_6-1 \end{pmatrix}. \tag{154}$$

We can apply Fubini's theorem [4, Prop.2.13] and Corollary 3.5 to perform the integral over x_1. After renaming the variable x_2 by x, we express $\Phi^{\Xi_{4_1}}(\hbar)$ by a one-dimensional formal Gaussian integral

$$\Phi^{\Xi_{4_1}}(\hbar) \;=\; e^{\frac{\hbar}{6}} \Big\langle \exp\Big(\frac{x}{2}\hbar^{\frac12}\Big)\psi_\hbar(x,\zeta_6)^2 \Big\rangle_{x,2\zeta_6-1}. \tag{155}$$

Using the definition of $\psi_\hbar$ from Equation (1) and expanding to $O(\hbar^{5/2})$, we obtain that

$$\begin{aligned}
&\exp\Big(\frac{x}{2}\hbar^{\frac12}\Big)\psi_\hbar(x,\zeta_6)^2 \\
&= 1+\Big(\frac13 x^3+\Big(\zeta_6-\frac12\Big)x\Big)\hbar^{1/2}+\Big(\frac{1}{18}x^6+\Big(\frac12\zeta_6-\frac14\Big)x^4-\frac78 x^2+\Big(-\frac16\zeta_6+\frac16\Big)\Big)\hbar \\
&\quad+\Big(\frac{1}{162}x^9+\Big(\frac19\zeta_6-\frac{1}{18}\Big)x^7-\frac12 x^5+\Big(-\frac{73}{72}\zeta_6+\frac{77}{144}\Big)x^3+\Big(\frac{1}{12}\zeta_6+\frac14\Big)x\Big)\hbar^{3/2} \\
&\quad+\Big(\frac{1}{1944}x^{12}+\Big(\frac{5}{324}\zeta_6-\frac{5}{648}\Big)x^{10}-\frac{37}{288}x^8+\Big(-\frac{1337}{2160}\zeta_6+\frac{1357}{4320}\Big)x^6 \\
&\quad+\Big(\frac{1}{24}\zeta_6+\frac{1027}{1152}\Big)x^4+\Big(\frac{23}{48}\zeta_6-\frac{5}{16}\Big)x^2-\frac{1}{72}\zeta_6\Big)\hbar^2+O(\hbar^{5/2})\,.
\end{aligned} \tag{156}$$

Noting that $2\zeta_6-1=\sqrt{-3}$, we can then evaluate Equation (155) with Equation (20) to obtain that

$$e^{-\frac{\hbar}{4}}\Phi^{\Xi_{4_1}}(\hbar) \;=\; 1+\frac{11}{72\sqrt{-3}}\hbar+\frac{697}{2(72\sqrt{-3})^2}\hbar^2+O(\hbar^4)\,. \tag{157}$$

This is in agreement with computations in [11, 12, 21]. The one-dimensional formal Gaussian integral (155) gives an effective computation of the series $\Phi^{\Xi_{4_1}}(\hbar)$. Indeed, using a `pari-gp` program one can compute one hundred coefficients in a few seconds and two hundred coefficients in a few minutes, the first few of them are given by

$$\begin{aligned}
&e^{-\frac{\hbar}{4}}\Phi^{\Xi_{4_1}}(\hbar) \\
&= 1-\frac{11}{216}\sqrt{-3}\,\hbar-\frac{697}{31104}\hbar^2+\frac{724351}{100776960}\sqrt{-3}\,\hbar^3+\frac{278392949}{29023764480}\hbar^4-\frac{244284791741}{43883931893760}\sqrt{-3}\,\hbar^5 \\
&\quad-\frac{1140363907117019}{94789292890521600}\hbar^6+\frac{212114205337147471}{20474487264352665600}\sqrt{-3}\,\hbar^7+\frac{36736284422996813155 7}{117933046642671353856 00}\hbar^8 \\
&\quad-\frac{44921192873529779078383921}{126094013470344211542835 2000}\sqrt{-3}\,\hbar^9-\frac{3174342130562495575602143407}{2310959374147399367912325 1200}\hbar^{10}+O(\hbar^{11})\,.
\end{aligned} \tag{158}$$

Similar to the case of the 4_1, one can obtain one-dimensional formal Gaussian integrals for the next two simplest hyperbolic knots, the 5_2 and the $(-2,3,7)$-pretzel knot, whose details we omit.

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Appendix A. Complements on the Fourier transform

In this appendix, we give the omitted details in the last step of the proof of Theorem 3.4. They are an affine change of coordinates, followed by the corresponding computation of the formal Gaussian integration.

The proof requires a version of formal Gaussian integration where the symmetric matrix $\Lambda_\hbar \in \mathrm{GL}_N(\mathbb{Q}(z)[x][\![\hbar^{1/2}]\!])$ depends on $\hbar$, such that Λ_0 is invertible. In this case, for an integrable function $f_\hbar(x,z) \in \mathbb{Q}(z)[x][\![\hbar^{1/2}]\!]$, we define

$$\begin{aligned}\langle\!\langle f_\hbar(x,z)\rangle\!\rangle_{x,\Lambda_\hbar} \;&:=\; \sqrt{\frac{\det(\Lambda_\hbar)}{\det(\Lambda_0)}}\Big\langle \exp\big(-\tfrac{1}{2}x^t(\Lambda_\hbar-\Lambda_0)x\big)\, f_\hbar(x,z)\Big\rangle_{x,\Lambda_0} \\ &=\; \frac{\int e^{-\frac{1}{2}x^t\Lambda(\hbar)\,x} f_\hbar(x,z)\,dx}{\int e^{-\frac{1}{2}x^t\Lambda(\hbar)\,x}\,dx} \in \mathbb{Q}(z)[\![\hbar]\!]\,.\end{aligned} \tag{159}$$

This version of formal Gaussian integration satisfies the properties of Lemmas 3.1 and 3.2.

We use Equations (34) and Equation (37) to obtain that

$$\psi_\hbar(x,z) \;=\; e^{-\frac{\hbar}{24}} C_\hbar(x,z)\,\Big\langle\!\!\Big\langle \exp\Big(\frac{y}{2}\hbar^{\frac{1}{2}}\Big)\,\psi_\hbar\Big(y,\frac{1}{1-ze^{x\hbar^{1/2}}}\Big)\Big\rangle\!\!\Big\rangle_{y,1-z^{-1}e^{-x\hbar^{1/2}}}. \tag{160}$$

Lemma 3.2 implies that

$$\begin{aligned}\psi_\hbar(x,z) \;=\; & C_\hbar(x,z)\,\exp\Big(-\frac{\hbar}{24}-\frac{\hbar^{1/2}}{2}(a+x)\Big) \\ & \times\Big\langle \exp\Big(\frac{y}{2}\hbar^{\frac{1}{2}}+\Big(\frac{1}{ze^{x\hbar^{1/2}}}-\frac{1}{z}\Big)\frac{y^2}{2}\Big)\,\psi_\hbar\Big(y,\frac{1}{1-ze^{x\hbar^{1/2}}}\Big)\Big\rangle_{y,1-z^{-1}},\end{aligned} \tag{161}$$

where $a = a_\hbar(x,z) \in \mathbb{Q}(z)[x][\![\hbar^{1/2}]\!]$ is given by

$$a \;:=\; \frac{1}{\hbar^{1/2}}\log\Big(\frac{1-z}{1-ze^{x\hbar^{1/2}}}\Big) \in \mathbb{Q}(z)[x][\![\hbar^{1/2}]\!]\,. \tag{162}$$

Similar to Equation (34) we write

$$\psi_\hbar\Big(y,\frac{1}{1-ze^{x\hbar^{1/2}}}\Big) \;=\; \exp\Big(A_0-(a(1-z^{-1})+x)y-\Big(\frac{1}{ze^{x\hbar^{1/2}}}-\frac{1}{z}\Big)\frac{y^2}{2}\Big)\,\psi_\hbar\Big(y+a,\frac{1}{1-z}\Big) \tag{163}$$

where $A_0 = A_{0,\hbar}(x,z) \in \frac{1}{\hbar}\mathbb{Q}(z)[x][\![\hbar^{1/2}]\!]$ is given by

$$\begin{aligned} A_0 \;&=\; \frac{1}{2}\Big(\log\Big(\frac{-ze^{x\hbar^{1/2}}}{1-ze^{x\hbar^{1/2}}}\Big) - \log\Big(\frac{-z}{1-z}\Big)\Big) + \frac{1}{\hbar}\Big(\mathrm{Li}_2\Big(\frac{1}{1-ze^{x\hbar^{1/2}}}\Big) - \mathrm{Li}_2\Big(\frac{1}{1-z}\Big)\Big) \\ &\quad + \frac{a^2}{2z} + \frac{a}{\hbar^{1/2}}\log\Big(\frac{-z}{1-z}\Big) \\ &=\; \frac{1}{2}a\hbar^{\frac{1}{2}} + \frac{1}{2}x\hbar^{\frac{1}{2}} + \frac{1}{\hbar}\Big(\mathrm{Li}_2\Big(\frac{1}{1-ze^{x\hbar^{1/2}}}\Big) - \mathrm{Li}_2\Big(\frac{1}{1-z}\Big)\Big) + \frac{a^2}{2z} + \frac{a}{\hbar^{1/2}}\log\Big(\frac{-z}{1-z}\Big). \end{aligned} \tag{164}$$

Then Equation (161) can be written as

$$\begin{aligned} \psi_\hbar(x,z) \;&=\; C_\hbar(x,z)\,\exp\Big(-\frac{\hbar}{24} - \frac{\hbar^{1/2}}{2}(a+x) + A_0\Big) \\ &\qquad \times \Big\langle \exp\Big(\frac{y}{2}\hbar^{\frac{1}{2}} - (a(1-z^{-1}+x)y)\Big)\,\psi_\hbar\Big(y+a, \frac{1}{1-z}\Big)\Big\rangle_{y,1-z^{-1}}. \end{aligned} \tag{165}$$

We make the change of variables

$$w \mapsto w - a + \frac{xz}{1-z} \tag{166}$$

and using Equation (23) of Lemma 3.1, we obtain that

$$\begin{aligned} \psi_\hbar(x,z) \;&=\; C_\hbar(x,z)\,\exp\Big(-\frac{\hbar}{24} - \frac{\hbar^{1/2}}{2}(a+x) + A_0 + \frac{a^2}{2} - \frac{a^2}{2z} + ax + \frac{x^2}{2(1-z^{-1})} - \frac{\hbar^{1/2}}{2}a\Big) \\ &\qquad \times \Big\langle \exp\Big(\frac{\hbar^{1/2}}{2}\Big(y + \frac{xz}{1-z}\Big)\Big)\,\psi_\hbar\Big(y+\frac{xz}{1-z}, \frac{1}{1-z}\Big)\Big\rangle_{y,1-z^{-1}}. \end{aligned} \tag{167}$$

Hence, it remains to show that

$$1 \;=\; C_\hbar(x,z)\,\exp\Big(-\frac{\hbar^{1/2}}{2}(a+x) + A_0 + \frac{a^2}{2} - \frac{a^2}{2z} + ax + \frac{x^2}{2(1-z^{-1})} - \frac{\hbar^{1/2}}{2}a\Big). \tag{168}$$

In other words, using the definitions of $C_\hbar(x,z)$ from Equation (34) and A_0 from Equation (164) it suffices to prove that

$$\begin{aligned} 0 \;&=\; \frac{1}{\hbar}\Big(\mathrm{Li}_2\Big(\frac{1}{1-ze^{x\hbar^{1/2}}}\Big) - \mathrm{Li}_2\Big(\frac{1}{1-z}\Big)\Big) + \frac{1}{\hbar}\big(\mathrm{Li}_2(z) - \mathrm{Li}_2\big(ze^{x\hbar^{1/2}}\big)\big) \\ &\quad + \frac{a}{\hbar^{1/2}}\log\Big(\frac{-z}{1-z}\Big) - \frac{x}{\hbar^{1/2}}\log(1-z) + \frac{a^2}{2} + ax. \end{aligned} \tag{169}$$

With the transformation formula of the dilogarithm

$$\mathrm{Li}_2\Big(\frac{1}{1-z}\Big) \;=\; \mathrm{Li}_2(z) - \frac{\pi^2}{3} + \log(z)\log(1-z) - \frac{1}{2}\log^2(z-1) \tag{170}$$

the right hand side of the previous equation is given by

$$\begin{aligned}&\frac{1}{\hbar}\Big(\log\left(ze^{x\hbar^{1/2}}\right)\log\left(1-ze^{x\hbar^{1/2}}\right)-\frac{1}{2}\log^2\left(ze^{x\hbar^{1/2}}-1\right)\\&\qquad-\log(z)\log(1-z)+\frac{1}{2}\log^2(z-1)\Big)\\&+\frac{a}{\hbar^{1/2}}\log\left(\frac{-z}{1-z}\right)-\frac{x}{\hbar^{1/2}}\log(1-z)+\frac{a^2}{2}+ax.\end{aligned}\tag{171}$$

With $x\hbar^{1/2}=\log\left(e^{x\hbar^{1/2}}\right)$ we compute

$$\begin{aligned}&\frac{1}{\hbar}\log\left(ze^{x\hbar^{1/2}}\right)\log\left(1-ze^{x\hbar^{1/2}}\right)-\frac{1}{\hbar}\log(z)\log(1-z)-\frac{x}{\hbar^{1/2}}\log(1-z)+ax\\&=\frac{1}{\hbar}\log\left(ze^{x\hbar^{1/2}}\right)\log\left(1-ze^{x\hbar^{1/2}}\right)-\frac{1}{\hbar}\log(z)\log(1-z)-\frac{1}{\hbar}\log(e^{x\hbar^{1/2}})\log(1-z)+\frac{a}{\hbar^{1/2}}\log\left(e^{x\hbar^{1/2}}\right)\\&=\frac{1}{\hbar}\log\left(ze^{x\hbar^{1/2}}\right)\Big(\log\left(1-ze^{x\hbar^{1/2}}\right)-\log(1-z)\Big)+\frac{a}{\hbar^{1/2}}\log\left(e^{x\hbar^{1/2}}\right)\\&=-\frac{a}{\hbar^{1/2}}\log\left(ze^{x\hbar^{1/2}}\right)+\frac{a}{\hbar^{1/2}}\log\left(e^{x\hbar^{1/2}}\right)\\&=-\frac{a}{\hbar^{1/2}}\log(z)\end{aligned}\tag{172}$$

so that Equation (171) becomes

$$\begin{aligned}&-\frac{1}{2\hbar}\log^2\left(ze^{x\hbar^{1/2}}-1\right)+\frac{1}{2\hbar}\log^2(z-1)+\frac{a}{\hbar^{1/2}}\log\left(\frac{-z}{1-z}\right)+\frac{a^2}{2}-\frac{a}{\hbar^{1/2}}\log(z)\\&=-\frac{1}{2\hbar}\log^2\left(ze^{x\hbar^{1/2}}-1\right)+\frac{1}{2\hbar}\log^2(z-1)-\frac{a}{\hbar^{1/2}}\log(z-1)+\frac{a^2}{2}\\&=-\frac{1}{2\hbar}\log^2\left(ze^{x\hbar^{1/2}}-1\right)+\frac{1}{2}\Big(\frac{1}{\hbar^{1/2}}\log(z-1)-a\Big)^2\\&=-\frac{1}{2\hbar}\log^2\left(ze^{x\hbar^{1/2}}-1\right)+\frac{1}{2\hbar}\log^2(ze^{x\hbar^{1/2}}-1)\\&=0,\end{aligned}\tag{173}$$

which completes the proof of the last step of Theorem 3.4.

Appendix B. Complements on the pentagon identity

In this appendix, we give the omitted details in the last step of the proof of Theorem 3.6. They are an affine change of coordinates, followed by the corresponding computation of the formal Gaussian integration.

Equations (49) and Equation (34) give

$$\psi_\hbar(x,z_1)\psi_\hbar(y,z_2) \;=\; e^{-\frac{\hbar}{24}}C_\hbar(x,z_1)C_\hbar(y,\hat{z}_1)\;\left\langle\!\left\langle\psi_\hbar(-w,\hat{z}_1\hat{z}_0^{-1})\psi_\hbar(w,\hat{z}_0)\psi_\hbar(-w,\hat{z}_2\hat{z}_0^{-1})\right\rangle\!\right\rangle_{w,\hat{\delta}}\,, \tag{174}$$

where

$$\begin{aligned}\hat{z}_1 &= z_1e^{x\hbar^{1/2}} & \hat{z}_2 &= z_2e^{y\hbar^{1/2}}\\ \hat{z}_0 &= \hat{z}_1+\hat{z}_1-\hat{z}_1\hat{z}_2, & \hat{\delta} &= \frac{(\hat{z}_2+\hat{z}_2-\hat{z}_1\hat{z}_2)^2}{\hat{z}_1\hat{z}_2(1-\hat{z}_1)(1-\hat{z}_2)}\,.\end{aligned}\tag{175}$$

Note that $\hat{z}_1$, $\hat{z}_2$, $\hat{z}_0$ and $\hat{\delta}$ are power series in $\hbar^{1/2}$ which, when evaluated at $\hbar = 0$, coincide with z_1, z_2, z_0 and δ given in Equations (46) and (47).

We apply Lemma 3.2 to obtain that

$$\begin{aligned}\psi_\hbar(x, z_1)\psi_\hbar(y, z_2) &= e^{-\frac{\hbar}{24}} C_\hbar(x, z_1) C_\hbar(y, \hat{z}_1) \exp\Big(\frac{1}{2}(\log\hat{\delta} - \log\delta)\Big) \\ &\qquad \times \Big\langle \exp\Big(\frac{w^2}{2}(\delta - \hat{\delta})\Big)\psi_\hbar(-w, \hat{z}_1\hat{z}_0^{-1})\psi_\hbar(w, \hat{z}_0)\psi_\hbar(-w, \hat{z}_2\hat{z}_0^{-1})\Big\rangle_{w,\delta}.\end{aligned} \tag{176}$$

where $a = a_\hbar(x, z) \in \mathbb{Q}(z)[x][\![\hbar^{1/2}]\!]$ is given by

$$a := \frac{1}{\hbar^{1/2}} \log(z_0 \hat{z}_0^{-1}). \tag{177}$$

We write similarly to Equation (34)

$$\begin{aligned}&\psi_\hbar(-w, \hat{z}_1\hat{z}_0^{-1})\psi_\hbar(w, \hat{z}_0)\psi_\hbar(-w, \hat{z}_2\hat{z}_0^{-1}) \\ &= \exp(A_0 + w((x + y) + 2a + \mathrm{Li}_0(z_1 z_0^{-1})(a + x) + \mathrm{Li}_0(z_0)a + \mathrm{Li}_0(z_2 z_0^{-1})(a + y)) + \frac{w^2}{2}(\hat{\delta} - \delta)) \\ &\quad \times \psi_\hbar(-w + a + x, \hat{z}_1\hat{z}_0^{-1})\psi_\hbar(w - a, \hat{z}_0)\psi_\hbar(-w + a + y, \hat{z}_2\hat{z}_0^{-1})\end{aligned} \tag{178}$$

where $A_0 = A_{0,\hbar}(x, z) \in \frac{1}{\hbar}\mathbb{Q}(z)[x][\![\hbar^{1/2}]\!]$ is given by

$$\begin{aligned}A_0 &:= \frac{1}{\hbar}(\mathrm{Li}_2(\hat{z}_1\hat{z}_0^{-1}) + \mathrm{Li}_2(\hat{z}_0) + \mathrm{Li}_2(\hat{z}_2\hat{z}_0^{-1}) - \mathrm{Li}_2(z_1 z_0^{-1}) + \mathrm{Li}_2(z_0) - \mathrm{Li}_2(z_2 z_0^{-1})) \\ &\quad + \frac{1}{2}(\log(1 - \hat{z}_1\hat{z}_0^{-1}) + \log(1 - \hat{z}_0) + \log(1 - \hat{z}_2\hat{z}_0^{-1}) \\ &\qquad - \log(1 - z_1\hat{z}_0^{-1}) + \log(1 - z_0) + \log(1 - z_2 z_0^{-1})) \\ &\quad + \frac{1}{\hbar^{1/2}}(\log(1 - z_1 z_0^{-1})(a + x) - \log(1 - z_0)a + \log(1 - z_2 z_0^{-1})(a + y)) \\ &\quad - \frac{1}{2}(\mathrm{Li}_0(z_1 z_0^{-1})(a + x)^2 + \mathrm{Li}_0(z_0)a^2 + \mathrm{Li}_0(z_2 z_0^{-1})(a + y)^2)\end{aligned} \tag{179}$$

Hence, $\psi_\hbar(x, z_1)\psi_\hbar(y, z_2)$ can be written as

$$\begin{aligned}&e^{-\frac{\hbar}{24}} C_\hbar(x, z_1) C_\hbar(y, \hat{z}_1) \exp\Big(\frac{1}{2}(\log(\hat{\delta}) - \log(\delta)) + A_0\Big) \\ &\times \Big\langle \exp\Big(w((u + v) + 2a + \mathrm{Li}_0(z_1 z_0^{-1})(a + u) + \mathrm{Li}_0(z_0)a + \mathrm{Li}_0(z_2 z_0^{-1})(a + v))\Big) \\ &\qquad \times \psi_\hbar(-w + a + x, \hat{z}_1\hat{z}_0^{-1})\psi_\hbar(w - a, \hat{z}_0)\psi_\hbar(-w, \hat{z}_2\hat{z}_0^{-1})\Big\rangle_{w+a+y,\delta}.\end{aligned} \tag{180}$$

With the change of variables

$$w \mapsto w + a + x + y - \frac{x z_1 + y z_2}{z_0} \tag{181}$$

combined with Equation (23) of Lemma 3.1, we obtain that

$$
\begin{aligned}
&e^{-\frac{\hbar}{24}} C_\hbar(x,z_1) C_\hbar(y,\hat z_1) \exp\Big(\frac{1}{2}(\log(\hat\delta)-\log(\delta))+A_0\Big)\\
&\exp\Big(\frac{\delta}{2}\Big(a+x+y-\frac{xz_2+yz_1}{z_0}\Big)^2+\Big(a+x+y-\frac{xz_2+yz_1}{z_0}\Big)(x+y+\delta a+\mathrm{Li}_0(z_1z_0^{-1})x+\mathrm{Li}_0(z_2z_0^{-1})y)\Big)\\
&\times\Big\langle \psi_\hbar\Big(-w-y+\frac{xz_2+yz_1}{z_0},\hat z_1\hat z_0^{-1}\Big)\psi_\hbar\Big(w+x+y+\frac{xz_2+yz_1}{z_0},\hat z_0\Big)\psi_\hbar\Big(-w-x+\frac{xz_2+yz_1}{z_0},\hat z_2\hat z_0^{-1}\Big)\Big\rangle_{w,\delta}.
\end{aligned}
\tag{182}
$$

Hence, in order to prove Equation (48) it remains to prove that the term in front of the bracket simplifies to $e^{-\frac{\hbar}{24}}$. For this, we use the definitions of $C_\hbar$ (35) and A_0 (179) to obtain

$$
\begin{aligned}
&\log(C_\hbar(x,z_1))+\log(C_\hbar(y,\hat z_1))+\frac{1}{2}(\log(\hat\delta)-\log(\delta))+A_0+\frac{\delta}{2}\Big(a+x+y-\frac{xz_2+yz_1}{z_0}\Big)^2\\
&+\Big(a+x+y-\frac{xz_2+yz_1}{z_0}\Big)(x+y+\delta a+\mathrm{Li}_0(z_1z_0^{-1})x+\mathrm{Li}_0(z_2z_0^{-1})y))\\
=&\frac{1}{\hbar}(-\mathrm{Li}_2(\hat z_1)+\mathrm{Li}_2(z_1)-\mathrm{Li}_2(\hat z_2)+\mathrm{Li}_2(z_2)+\mathrm{Li}_2(\hat z_1\hat z_0^{-1})-\mathrm{Li}_2(z_1z_0^{-1})\\
&\quad+\mathrm{Li}_2(\hat z_0^{-1})-\mathrm{Li}_2(z_0)+\mathrm{Li}_2(\hat z_2\hat z_0^{-1})-\mathrm{Li}_2(z_2z_0^{-1}))\\
&+\frac{1}{2}(-\log(1-\hat z_1)+\log(1-z_1)-\log(1-\hat z_2)+\log(1-z_2)+\log\hat\delta-\log\delta\\
&\quad+\log(1-\hat z_1\hat z_0^{-1})-\log(1-z_1z_0^{-1})+\log(1-\hat z_0)-\log(1-z_0)\\
&\quad+\log(1-\hat z_2\hat z_0^{-1})-\log(1-z_2z_0^{-1}))\\
&+\frac{1}{\hbar^{1/2}}(-\log(1-z_1)x-\log(1-z_2)y+(x+y)(\log(z_0)-\log(\hat z_o))\\
&\quad+\log(1-z_1z_0^{-1})(a+x)-\log(1-z_0)a+\log(1-z_2z_0^{-1})(a+y))\\
&+\frac{a^2}{2}(\delta-\mathrm{Li}_0(z_1z_0^{-1})-\mathrm{Li}_0(z_0)-\mathrm{Li}_0(z_2z_0^{-1}))\\
&+\Big(x+y-\frac{xz_2+yz_2}{z_0}\Big)\Big(x+y+\mathrm{Li}_0(z_1z_0^{-1})x+\mathrm{Li}_0(z_2z_0^{-1})y-\frac{\delta}{2}\Big(x+y-\frac{xz_2+yz_2}{z_0}\Big)\Big)\\
&\quad-\mathrm{Li}_0(z_1z_0^{-1})\frac{x^2}{2}-\mathrm{Li}_0(z_2z_0^{-1})\frac{y^2}{2}.
\end{aligned}
\tag{183}
$$

Inserting the definitions of δ (47) and $\hat\delta$ (175) and using the relations

$$
\begin{aligned}
1-z_1z_0^{-1} &= z_2(1-z_1)z_0^{-1},\\
1-z_2z_0^{-1} &= z_1(1-z_2)z_0^{-1},\\
1-z_0 &= (1-z_1)(1-z_2),
\end{aligned}
\tag{184}
$$

and similar ones for $\hat z_1, \hat z_2$ and $\hat z_0$ we obtain that the terms

$$
\begin{aligned}
&\frac{1}{2}(-\log(1-\hat z_1)+\log(1-z_1)-\log(1-\hat z_2)+\log(1-z_2)+\log\hat\delta-\log\delta\\
&\quad+\log(1-\hat z_1\hat z_0^{-1})-\log(1-z_1z_0^{-1})+\log(1-\hat z_0)-\log(1-z_0)\\
&\quad+\log(1-\hat z_2\hat z_0^{-1})-\log(1-z_2z_0^{-1}))
\end{aligned}
\tag{185}
$$

vanish. Furthermore, we have

$$\delta - \mathrm{Li}_0(z_1 z_0^{-1}) - \mathrm{Li}_0(z_0) - \mathrm{Li}_0(z_2 z_0^{-1}) \;=\; 2 \tag{186}$$

as well as

$$\begin{aligned}&\Big(x + y - \frac{xz_2 + yz_2}{z_0}\Big)\Big(x + y + \mathrm{Li}_0(z_1 z_0^{-1})x + \mathrm{Li}_0(z_2 z_0^{-1})y - \frac{\delta}{2}\Big(x + y - \frac{xz_2 + yz_2}{z_0}\Big)\Big)\\ &- \mathrm{Li}_0(z_1 z_0^{-1})\frac{x^2}{2} - \mathrm{Li}_0(z_2 z_0^{-1})\frac{y^2}{2} \;=\; xy.\end{aligned} \tag{187}$$

Therefore, Equation (183) simplifies to

$$\begin{aligned}&\frac{1}{\hbar}(-\mathrm{Li}_2(\hat{z}_1) + \mathrm{Li}_2(z_1) - \mathrm{Li}_2(\hat{z}_2) + \mathrm{Li}_2(z_2) + \mathrm{Li}_2(\hat{z}_1\hat{z}_0^{-1}) - \mathrm{Li}_2(z_1 z_0^{-1})\\ &\qquad + \mathrm{Li}_2(\hat{z}_0^{-1}) - \mathrm{Li}_2(z_0) + \mathrm{Li}_2(\hat{z}_2\hat{z}_0^{-1}) - \mathrm{Li}_2(z_2 z_0^{-1}))\\ &+ \frac{1}{\hbar^{1/2}}(-\log(1 - z_1)x - \log(1 - z_2)y + (x + y)(\log(z_0) - \log(\hat{z}_o))\\ &\qquad + \log(1 - z_1 z_0^{-1})(a + x) - \log(1 - z_0)a + \log(1 - z_2 z_0^{-1})(a + y))\\ &+ a^2 + xy.\end{aligned} \tag{188}$$

Using the definition of a and the relations from Equation (184) we compute

$$\begin{aligned}&\frac{1}{\hbar^{1/2}}(-\log(1 - z_1)x - \log(1 - z_2)y + (x + y)(\log(z_0) - \log(\hat{z}_o))\\ &\qquad + \log(1 - z_1 z_0^{-1})(a + x) - \log(1 - z_0)a + \log(1 - z_2 z_0^{-1})(a + y))\\ &+ a^2 + xy\\ =\;& \frac{1}{\hbar^{1/2}}(x(\log(z_2) - \log(\hat{z}_0)) + y(\log(z_1) - \log(\hat{z}_0))\\ &+ \frac{1}{\hbar}(\log(z_0)(\log(z_1) + \log(z_2)) + \log^2(z_0) - \log(\hat{z}_0)(\log(z_1) + \log(z_2)))\\ =\;& \frac{1}{\hbar}(\log(z_2)\log(z_0) + \log(z_1)\log(z_0) - \log(z_1)\log(z_2)\\ &\qquad - \log(\hat{z}_2)\log(\hat{z}_0) + \log(\hat{z}_1)\log(\hat{z}_2) - \log(\hat{z}_1)\log(\hat{z}_0)).\end{aligned} \tag{189}$$

Using the 5–term relation of the dilogarithm we obtain that the expression in Equation (188)

$$\begin{aligned}&\mathrm{Li}_2(z_1) + \mathrm{Li}_2(z_2) - \mathrm{Li}_2(z_1 z_0^{-1}) - \mathrm{Li}_2(z_0) - \mathrm{Li}_2(z_2 z_0^{-1}) + \log(z_2)\log(z_0) + \log(z_1)\log(z_0) - \log(z_1)\log(z_2)\\ &- \mathrm{Li}_2(\hat{z}_1) - \mathrm{Li}_2(\hat{z}_2) + \mathrm{Li}_2(\hat{z}_1\hat{z}_0^{-1}) + \mathrm{Li}_2(\hat{z}_0^{-1}) + \mathrm{Li}_2(\hat{z}_2\hat{z}_0^{-1}) - \log(\hat{z}_2)\log(\hat{z}_0) - \log(\hat{z}_1)\log(\hat{z}_0) + \log(\hat{z}_1)\log(\hat{z}_2).\end{aligned} \tag{190}$$

vanishes. In particular, the terms in front of the bracket in Equation (182) simplify to $e^{-\frac{\hbar}{24}}$ which concludes the proof of the last step of Theorem 3.6.

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