

# GENERALIZED RANK DEVIATIONS FOR OVERPARTITIONS

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ABSTRACT. We prove formulas for generalized rank deviations for overpartitions. These formulas are in terms of Appell–Lerch series and sums of quotients of theta functions and extend work of Lovejoy and the second author. As an application, we compute a dissection.

## 1. INTRODUCTION

A partition of a natural number  $n$  is a non-increasing sequence of positive integers whose sum is  $n$ . An overpartition of  $n$  is a partition in which the first occurrence of each distinct part may be overlined. For example, the partitions and overpartitions of 4 are

$$4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1$$

and

$$4, \overline{4}, 3 + 1, \overline{3} + 1, 3 + \overline{1}, \overline{3} + \overline{1}, 2 + 2, \overline{2} + 2, 2 + 1 + 1, \overline{2} + 1 + 1, 2 + \overline{1} + 1, \\ \overline{2} + \overline{1} + 1, 1 + 1 + 1 + 1, \overline{1} + 1 + 1 + 1,$$

respectively. For a curated survey which highlights the importance of overpartitions in  $q$ -series, number theory and algebraic combinatorics, see [10]. The rank of a partition  $\lambda$  is the largest part  $\ell(\lambda)$  minus the number of parts  $\#(\lambda)$ . For overpartitions, one can consider two statistics, the rank or the  $M_2$ -rank which is defined by [19]

$$M_2\text{-rank}(\pi) = \left\lfloor \frac{\ell(\pi)}{2} \right\rfloor - \#(\pi) + \#(\pi_o) - \chi(\pi)$$

where  $\pi_o$  is the subpartition consisting of the odd non-overlined parts and

$$\chi(\pi) := \chi(\text{the largest part of } \pi \text{ is odd and non-overlined}).$$

Throughout, we use the standard notation  $\chi(X) := 1$  if  $X$  is true and 0 if  $X$  is false. A recent topic of interest for these combinatorial objects and their enumerative data is the study of rank deviations [16, 20]. For integers  $0 \leq a \leq M$  where  $M \geq 2$ , consider

$$\overline{D}(a, M) := \sum_{n \geq 0} \left( \overline{N}(a, M, n) - \frac{\overline{p}(n)}{M} \right) q^n \tag{1.1}$$

and

$$\overline{D}_2(a, M) := \sum_{n \geq 0} \left( \overline{N}_2(a, M, n) - \frac{\overline{p}(n)}{M} \right) q^n \tag{1.2}$$

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where  $\overline{N}(a, M, n)$  denotes the number of overpartitions of  $n$  with rank congruent to  $a$  modulo  $M$ ,  $\overline{N}_2(a, M, n)$  denotes the number of overpartitions of  $n$  with  $M_2$ -rank congruent to  $a$  modulo  $M$  and  $\overline{p}(n)$  is the number of overpartitions of  $n$ . Some impetuses for finding explicit formulas for the rank deviations (1.1) and (1.2) are to correct the literature [20, Remark 1.6], provide a general framework in which one can recover all known rank difference identities for overpartitions [20, Section 4] and prove the modularity of rank generating functions in arithmetic progressions (e.g., [16, Section 7]) thereby generalizing [8, 11, 21].

For an integer  $d \geq 1$ , consider the generating function

$$\begin{aligned} \mathcal{O}_d(z; q) &:= \frac{(-q)_\infty}{(q)_\infty} \left( 1 + 2 \sum_{n \geq 1} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+dn}}{(1-zq^{dn})(1-z^{-1}q^{dn})} \right) \\ &=: \sum_{\substack{m \in \mathbb{Z} \\ n \geq 0}} \overline{N}_d(m, n) z^m q^n \end{aligned} \quad (1.3)$$

where

$$(x)_\infty = (x; q)_\infty := \prod_{k=0}^{\infty} (1 - xq^k).$$

The  $d = 1$  case of (1.3) gives the two-variable generating function for the rank for overpartitions [18] while the  $d = 2$  case yields the two-variable generating function for the  $M_2$ -rank for overpartitions [19]. The mock and quantum modular properties of (1.3) were the focus of [12, 17]. Motivated by [12, 17, 20], our aim is to find explicit formulas for the generalized rank deviations for overpartitions

$$\overline{D}_d(a, M) := \sum_{n \geq 0} \left( \overline{N}_d(a, M, n) - \frac{\overline{p}(n)}{M} \right) q^n \quad (1.4)$$

where

$$\overline{N}_d(a, M, n) := \sum_{k \equiv a \pmod{M}} \overline{N}_d(k, n). \quad (1.5)$$

To state our main results, we require some setup. Consider the Appell–Lerch series

$$m(x, q, z) := \frac{1}{j(z; q)} \sum_{r \in \mathbb{Z}} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1 - q^{r-1} xz}. \quad (1.6)$$

Here,  $x$ ,  $q$  and  $z$  are non-zero complex numbers with  $|q| < 1$ , neither  $z$  nor  $xz$  is an integral power of  $q$  and

$$j(z; q) := (z)_\infty (q/z)_\infty (q)_\infty = \sum_{n \in \mathbb{Z}} (-1)^n q^{\binom{n}{2}} z^n. \quad (1.7)$$

Let

$$\Delta(x, z_1, z_0; q) := \frac{z_0 J_1^3 j(z_1/z_0; q) j(xz_0 z_1; q)}{j(z_0; q) j(z_1; q) j(xz_0; q) j(xz_1; q)} \quad (1.8)$$

and

$$\Psi_k^n(x, z, z'; q) := -\frac{x^k z^{k+1} J_{n_2}^3}{j(z; q) j(z'; q^{n^2})} \sum_{t=0}^{n-1} \frac{q^{\binom{t+1}{2} + kt} (-z)^t j(-q^{\binom{n+1}{2} + nk + nt} (-z)^n / z', q^{nt} (xz)^n z'; q^{n^2})}{j(-q^{\binom{n}{2} - nk} (-x)^n z', q^{nt} (xz)^n; q^{n^2})} \quad (1.9)$$

where  $J_m := (q^m; q^m)_\infty$  and  $j(z_1, z_2; q) := j(z_1; q)j(z_2; q)$ . We use the term “generic” to mean that the parameters do not cause poles in the Appell–Lerch series or in the quotients of theta functions. Finally, for odd  $d \geq 1$  and generic  $z, z_0$  and  $z' \in \mathbb{C}^*$ , let

$$\begin{aligned} \Lambda(d, z, z_0, z') := & (-1)^{\frac{d+1}{2}} q^{-\frac{(d-1)^2}{4}} z^{\frac{d-1}{d}} \left( \Psi_{\frac{d-1}{2}}^d(z^{-\frac{2}{d}} q^d, z_0, z'; q^2) \right. \\ & \left. + \frac{1}{d} \sum_{t=0}^{d-1} \zeta_d^{-t} \Delta(\zeta_d^{-2t} z^{-\frac{2}{d}} q^d, \zeta_d^t z^{\frac{1}{d}} q^{-\frac{d-1}{2}}, z_0; q^2) \right). \end{aligned} \quad (1.10)$$

Here and throughout,  $\zeta_M$  denotes a primitive root of unity of order  $M$ .

We can now state our two main results in which explicit formulas are given for the pair of deviations  $\overline{D}_d(a, M) + \overline{D}_d(a-1, M)$ . Theorem 1.1 addresses the case of odd  $d$  and is slightly more involved with the appearance of the additional term (1.10). Theorem 1.2 treats the case of even  $d$  and is cleaner. As discussed in Section 4, there is no loss in generality in computing pairs of deviations. Our results can be used to find a formula for any single  $\overline{D}_d(a, M)$ .

**Theorem 1.1.** *Let  $d \geq 1$  be an odd integer and  $2 \leq a \leq M$ . For generic  $z', z''$  and  $z_0 \in \mathbb{C}^*$ , we have the following generating functions:*

(i) *If  $a$  and  $M$  are even, then*

$$\begin{aligned} \overline{D}_d(a, M) + \overline{D}_d(a-1, M) = & \chi(a = M) \\ & + 2(-1)^{\frac{a}{2}} q^{-\frac{a^2}{4} + \frac{a}{2}(1-d^2)} m((-1)^{\frac{M}{2}+1} q^{\frac{M^2}{4} - \frac{aM}{2} + \frac{M}{2}(1-d^2)}, q^{\frac{M^2}{2}}, z') \\ & - 2q^{-d^2} \Psi_{\frac{a}{2}-1}^{\frac{M}{2}}(q^{-d^2}, -1, z'; q^2) \\ & + \frac{2}{M} \sum_{j=1}^M \zeta_{\frac{M}{2}}^{-aj} (1 - \zeta_M^j) \Lambda(d, \zeta_M^j, z_0, -1). \end{aligned} \quad (1.11)$$

(ii) *If  $a$  is even and  $M$  is odd, then*

$$\begin{aligned} \overline{D}_d(a, M) + \overline{D}_d(a-1, M) = & 2(-1)^{\frac{a}{2}} q^{-d^2(\frac{2M-a}{2})^2} m(q^{d^2 M(a-M)}, q^{2d^2 M^2}, z') \\ & + 2(-1)^{\frac{M+1-a}{2}} q^{-d^2(\frac{M+1-a}{2})^2} m(q^{d^2 M(a-1)}, q^{2d^2 M^2}, z'') \\ & - 2\Psi_{\frac{2M-a}{2}}^M(q^{d^2}, -1, z'; q^{2d^2}) + 2\Psi_{\frac{M+1-a}{2}}^M(q^{d^2}, -1, z''; q^{2d^2}) \\ & + \frac{2}{M} \sum_{j=1}^{M-1} \zeta_M^{-aj} (1 - \zeta_M^j) \Lambda(d, \zeta_M^j, z_0, -1). \end{aligned} \quad (1.12)$$

(iii) If  $a$  is odd and  $M$  is odd, then

$$\begin{aligned}
\overline{D}_d(a, M) + \overline{D}_d(a-1, M) &= \chi(a = M) \\
&\quad - 2(-1)^{\frac{M-a}{2}} q^{-d^2(\frac{M-a}{2})^2} m(q^{d^2 M a}, q^{2d^2 M^2}, z') \\
&\quad + 2(-1)^{\frac{a+1}{2}} q^{-d^2(\frac{2M-a+1}{2})^2} m(q^{d^2 M(a-M-1)}, q^{2d^2 M^2}, z'') \\
&\quad - 2\Psi_{\frac{M-a}{2}}^M(q^{d^2}, -1, z'; q^{2d^2}) + 2\Psi_{\frac{2M+1-a}{2}}^M(q^{d^2}, -1, z''; q^{2d^2}) \\
&\quad + \frac{2}{M} \sum_{j=1}^{M-1} \zeta_M^{-aj} (1 - \zeta_M^j) \Lambda(d, \zeta_M^j, z_0, -1).
\end{aligned} \tag{1.13}$$

**Theorem 1.2.** Let  $d \geq 2$  be an even integer and  $1 \leq a \leq M-1$ . For generic  $z', z'' \in \mathbb{C}^*$ , we have the generating function

$$\begin{aligned}
\overline{D}_d(a, M) + \overline{D}_d(a-1, M) &= \chi(a = 1) + 2(-1)^{\frac{da}{2}} q^{-\frac{d^2 a^2}{4}} m((-1)^{1+\frac{dM}{2}} q^{\frac{d^2}{4}(M^2-2Ma)}, q^{\frac{d^2 M^2}{2}}, z') \\
&\quad + 2(-1)^{\frac{d}{2}(a-1)+1} q^{-\frac{d^2}{4}(a^2-2a+1)} m((-1)^{1+\frac{dM}{2}} q^{\frac{d^2}{4}(M^2-2M(a-1))}, q^{\frac{d^2 M^2}{2}}, z'') \\
&\quad + 2\Psi_a^M((-1)^{\frac{d}{2}+1} q^{\frac{d^2}{4}}, -1, z'; q^{\frac{d^2}{2}}) - 2\Psi_{a-1}^M((-1)^{\frac{d}{2}+1} q^{\frac{d^2}{4}}, -1, z''; q^{\frac{d^2}{2}}) \\
&\quad + \frac{2}{M} (-1)^{\frac{d}{2}} q^{-\frac{d^2}{4}} \sum_{j=1}^{M-1} \zeta_M^{-aj} (1 - \zeta_M^j) \Psi_0^{\frac{d}{2}}(\zeta_M^{\frac{2j}{d}} q^{1-d}, q, -1; q^2).
\end{aligned} \tag{1.14}$$

**Remark 1.3.** From [17, page 1154] and (1.5), we have

$$\overline{N}_d(a, M, n) = \overline{N}_d(M-a, M, n)$$

and so by (1.4)

$$\overline{D}_d(a, M) = \overline{D}_d(M-a, M). \tag{1.15}$$

If  $d$  and  $a$  are odd and  $M$  is even, then  $\overline{D}_d(a, M) + \overline{D}_d(a-1, M)$  is computed from Theorem 1.1 (i) using the fact that

$$\overline{D}_d(a, M) + \overline{D}_d(a-1, M) = \overline{D}_d(M-a+1, M) + \overline{D}_d(M-a, M)$$

which follows from (1.15).

**Remark 1.4.** Let  $n \in \mathbb{Z}$ . If we take  $d = 1$  and  $z_0 = -1$  in Theorem 1.1 and use

$$j(q^n; q) = 0, \tag{1.16}$$

then one recovers [20, Theorem 1.1]. If we let  $d = 2$  in Theorem 1.2 and employ [15, Eqs. (2.2a) and (2.2b)]

$$j(q^n x; q) = (-1)^n q^{-\binom{n}{2}} x^{-n} j(x; q) \tag{1.17}$$

and

$$j(x; q) = j(q/x; q) = -x j(x^{-1}; q), \tag{1.18}$$

then we obtain [20, Theorem 1.2].

Although it is not our primary focus, Theorems 1.1 and 1.2 in combination with the techniques in [16, Section 7] can be used to deduce the modularity of  $\overline{D}_d(a, M)$  in arithmetic progressions. This is a “higher-rank” overpartition analogue of the celebrated results of Bringmann and Ono [24, Section 7]. One can also obtain the mock modularity of  $\mathcal{O}_d(z; q)$  where  $z$  is a root of unity. The idea is to express  $\mathcal{O}_d(z; q)$  as a sum of pairs of rank deviations (e.g., see (5.5)). One then applies Theorem 1.1 or 1.2 to express  $\mathcal{O}_d(z; q)$  in terms of Appell–Lerch series and sums of theta quotients (e.g., see (5.6)). As it is well-known that specializations of Appell–Lerch series are mock theta functions [25], the mock modularity of  $\mathcal{O}_d(z; q)$  follows. Thus, our results imply a more explicit version of [17, Theorems 1.1 and 1.2]. Finally, Theorems 1.1 and 1.2 provide a general framework in which one can recover any dissection of  $\mathcal{O}_d(z; q)$  where  $z$  is a root of unity and any generalized rank difference identity. See Section 5 for an example.

The paper is organized as follows. In Section 2, we recall the required background on Appell–Lerch series and prove two crucial formulas (see Proposition 2.4) which express a normalized version of  $\mathcal{O}_d(z; q)$  in terms of Appell–Lerch series and (1.10) if  $d$  is odd and in terms of Appell–Lerch series and (1.9) if  $d$  is even. We also give some other key properties of (1.7) and establish 3-dissections of certain eta quotients, both of which will be beneficial in Section 5. In Section 3, we prove Theorems 1.1 and 1.2. In Section 4, we explain why there is no loss in generality in considering pairs of generalized rank deviations in Theorems 1.1 and 1.2. In Section 5, we provide an application of Theorem 1.1 by computing the 3-dissection of  $\mathcal{O}_3(\zeta_3; q)$  (cf. [17, Theorem 1.3]).

Finally, we comment that the combinatorial interpretation of  $\overline{N}_d(m, n)$  involves a certain weighted count of buffered Frobenius representations [22, Theorem 1.3]. It is still an open problem to find a proper combinatorial definition of the “ $M_d$ -rank” for overpartitions [22, Section 6].

## 2. PRELIMINARIES

**2.1. Appell–Lerch series and  $\overline{S}_d(z; q)$ .** We first state two key results concerning Appell–Lerch series. The first relates two such series with different generic parameters  $z_1$  and  $z_0$  [15, Theorem 3.3] while the second is an orthogonality result [15, Theorem 3.9]. Recall (1.6), (1.8) and (1.9).

**Lemma 2.1.** *For generic  $x, z_0$  and  $z_1 \in \mathbb{C}^*$ ,*

$$m(x, q, z_1) - m(x, q, z_0) = \Delta(x, z_1, z_0; q). \quad (2.1)$$

**Lemma 2.2.** *Let  $n$  and  $k$  be integers with  $0 \leq k < n$ . Then*

$$\sum_{t=0}^{n-1} \zeta_n^{-kt} m(\zeta_n^t x, q, z) = nq^{-\binom{k+1}{2}} (-x)^k m(-q^{\binom{n}{2}-nk} (-x)^n, q^{n^2}, z') + n\Psi_k^n(x, z, z'; q). \quad (2.2)$$

We also require [15, Eqs. (3.2b), (3.2e) and (3.3)]

$$m(x, q, z) = x^{-1} m(x^{-1}, q, z^{-1}), \quad (2.3)$$

$$m(x, q, z) = x^{-1} - x^{-1} m(qx, q, z), \quad (2.4)$$

$$m(q, q^2, -1) = \frac{1}{2} \quad (2.5)$$

and [15, Eq. (4.7)]

$$\frac{1}{j(q; q^2)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n^2+n}}{1 - xq^n} = -x^{-1} m(x^{-2}q, q^2, x). \quad (2.6)$$

In addition, we often use the following well-known fact. Let  $s \in \mathbb{Z}$ . Then

$$\sum_{j=0}^{n-1} \zeta_n^{sj} = \begin{cases} n & \text{if } s \equiv 0 \pmod{n}, \\ 0 & \text{otherwise.} \end{cases} \quad (2.7)$$

To explicitly compute the generalized rank deviations (1.4), we use the following formula. As the proof is similar to that of [20, Proposition 2.3], we omit it. Let

$$\overline{S}_d(z; q) := (1+z)\mathcal{O}_d(z; q). \quad (2.8)$$

**Proposition 2.3.** *We have*

$$\frac{1}{M} \sum_{j=1}^{M-1} \zeta_M^{-aj} \overline{S}_d(\zeta_M^j; q) = \overline{D}_d(a, M) + \overline{D}_d(a-1, M). \quad (2.9)$$

We now need to express  $\overline{S}_d(z; q)$  in terms of Appell–Lerch series and sums of theta quotients. Recall (1.10).

**Proposition 2.4.** *Let  $z, z_0$  and  $z' \in \mathbb{C}^*$  be generic. For  $d$  odd, we have*

$$\overline{S}_d(z; q) = (1-z) \left( 1 - 2m(z^{-2}q^{d^2}, q^{2d^2}, z') + 2\Lambda(d, z, z_0, z') \right). \quad (2.10)$$

*For  $d$  even, we have*

$$\overline{S}_d(z; q) = (1-z) \left( -1 + 2m((-1)^{\frac{d}{2}+1} z q^{\frac{d^2}{4}}, q^{\frac{d^2}{2}}, z') + 2(-1)^{\frac{d}{2}} z q^{-\frac{d^2}{4}} \Psi_0^{\frac{d}{2}}(z^{\frac{2}{d}} q^{1-d}, q, z'; q^2) \right). \quad (2.11)$$

*Proof.* By (1.3), the fact that

$$\frac{1}{(1-zq^{dn})(1-z^{-1}q^{dn})} = \frac{-z^2}{(1-z^2)(1-zq^{dn})} + \frac{1}{(1-z^2)(1-z^{-1}q^{dn})}$$

and some simplifications, we have

$$\mathcal{O}_d(z; q) = \frac{1-z}{1+z} \left( 1 + \frac{2z}{j(q; q^2)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n^2+dn}}{1-zq^{dn}} \right). \quad (2.12)$$

For  $d$  odd, we let  $n \rightarrow n - \frac{d-1}{2}$  in (2.12) to obtain

$$\mathcal{O}_d(z; q) = \frac{1-z}{1+z} \left( 1 + \frac{2z(-1)^{\frac{d-1}{2}} q^{-\frac{(d-1)^2}{4} - \frac{d-1}{2}}}{j(q; q^2)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n^2+n}}{1-zq^{dn - \frac{d(d-1)}{2}}} \right). \quad (2.13)$$

An application of

$$\frac{1}{1-x^d} = \frac{1}{d} \sum_{t=0}^{d-1} \frac{1}{1-\zeta_d^t x} \quad (2.14)$$

to (2.13) yields

$$\mathcal{O}_d(z; q) = \frac{1-z}{1+z} \left( 1 + \frac{2z(-1)^{\frac{d-1}{2}} q^{-\frac{(d-1)^2}{4} - \frac{d-1}{2}}}{dj(q; q^2)} \sum_{t=0}^{d-1} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n^2+n}}{1-\zeta_d^t z^{\frac{1}{d}} q^{-\frac{d-1}{2}+n}} \right). \quad (2.15)$$

Taking  $x = \zeta_d^t z^{\frac{1}{d}} q^{-\frac{d-1}{2}}$  in (2.6) turns (2.15) into

$$\mathcal{O}_d(z; q) = \frac{1-z}{1+z} \left( 1 - \frac{2z(-1)^{\frac{d-1}{2}} q^{-\frac{(d-1)^2}{4}} z^{-\frac{1}{d}}}{d} \sum_{t=0}^{d-1} \zeta_d^{-t} m(\zeta_d^{-2t} z^{-\frac{2}{d}} q^d, q^2, \zeta_d^t z^{\frac{1}{d}} q^{-\frac{d-1}{2}}) \right). \quad (2.16)$$

We now let  $x = \zeta_d^{-2t} z^{-\frac{2}{d}} q^d$ ,  $q \rightarrow q^2$  and  $z_1 = \zeta_d^t z^{\frac{1}{d}} q^{-\frac{d-1}{2}}$  in (2.1) to express (2.16) as

$$\mathcal{O}_d(z; q) = \frac{1-z}{1+z} \left( 1 - \frac{2z(-1)^{\frac{d-1}{2}} q^{-\frac{(d-1)^2}{4}} z^{-\frac{1}{d}}}{d} \left( \sum_{t=0}^{d-1} \zeta_d^{-t} m(\zeta_d^{-2t} z^{-\frac{2}{d}} q^d, q^2, z_0) + \sum_{t=0}^{d-1} \zeta_d^{-t} \Delta(\zeta_d^{-2t} z^{-\frac{2}{d}} q^d, \zeta_d^t z^{\frac{1}{d}} q^{-\frac{d-1}{2}}, z_0; q^2) \right) \right). \quad (2.17)$$

Finally, we apply (2.2) with  $n = d$ ,  $k = \frac{d-1}{2}$ ,  $q \rightarrow q^2$  and  $x = z^{-\frac{2}{d}} q^d$  to (2.17), simplify and appeal to (1.10). By (2.8), this yields (2.10).

For  $d$  even, we let  $n \rightarrow n - \frac{d}{2}$  in (2.12) to obtain

$$\mathcal{O}_d(z; q) = \frac{1-z}{1+z} \left( 1 + \frac{2z(-1)^{\frac{d}{2}} q^{-\frac{d^2}{4}}}{j(q; q^2)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n^2}}{1 - zq^{dn - \frac{d^2}{2}}} \right). \quad (2.18)$$

Using (2.14), (2.18) becomes

$$\mathcal{O}_d(z; q) = \frac{1-z}{1+z} \left( 1 + \frac{4z(-1)^{\frac{d}{2}} q^{-\frac{d^2}{4}}}{d} \sum_{t=0}^{\frac{d}{2}-1} m(\zeta_{\frac{d}{2}}^t z^{\frac{2}{d}} q^{1-d}, q^2, q) \right). \quad (2.19)$$

Applying (2.2) with  $n = \frac{d}{2}$ ,  $k = 0$ ,  $q \rightarrow q^2$  and  $x = z^{\frac{2}{d}} q^{1-d}$  to (2.19) yields

$$\mathcal{O}_d(z; q) = \frac{1-z}{1+z} \left( 1 + 2z(-1)^{\frac{d}{2}} q^{-\frac{d^2}{4}} m((-1)^{\frac{d}{2}+1} zq^{-\frac{d^2}{4}}, q^{\frac{d^2}{2}}, z') + 2(-1)^{\frac{d}{2}} zq^{-\frac{d^2}{4}} \Psi_0^{\frac{d}{2}}(z^{\frac{2}{d}} q^{1-d}, q, z'; q^2) \right).$$

By (2.4) and (2.8), (2.11) follows.  $\square$

**Remark 2.5.** If we take  $d = 1$  and  $z_0 = z' = -1$  in (2.10), then use (1.10) combined with (1.18) and the  $z_1 = -1$ ,  $z_0 = z$  case of (2.1), we recover [20, Eq. (3.1)]. If we take  $d = 2$  and  $z' = q$  in (2.11) and then use (1.9) and (1.16), we obtain [20, Eq. (3.3)].

**2.2. Identities with  $j(z; q)$  and 3-dissections.** In order to facilitate an application of Theorem 1.1 in Section 5, we begin with some identities, the first two of which in the next result follow from (1.7) and (1.18) while the rest appear in [15, Eqs. (2.2f), (2.4a), (2.4b), (2.4c) and (2.4f)].

Moreover, we frequently use without mention

$$\begin{aligned} j(q; q^2) &= \frac{J_1^2}{J_2}, & j(q; q^3) &= J_1, & j(q; q^6) &= \frac{J_1 J_6^2}{J_2 J_3}, \\ j(-1; q) &= 2 \frac{J_2^2}{J_1}, & j(-q; q^3) &= \frac{J_2 J_3^2}{J_1 J_6} & \text{and} & j(-q; q^6) = \frac{J_2^2 J_3 J_{12}}{J_1 J_4 J_6}. \end{aligned}$$

**Proposition 2.6.** *For generic  $x, y, z \in \mathbb{C}^*$ , we have*

$$j(xq^{-1}; q^2)j(x^{-1}q^2; q^2) = x^2 q^{-1} j(x^{-1}; q) \frac{J_2^2}{J_1}, \quad (2.20)$$

$$\frac{j(-x; q)}{j(-x^2 q; q^2)j(x; q)} = -\frac{j(-x^{-1}; q)}{j(-x^{-2} q; q^2)j(x^{-1}; q)}, \quad (2.21)$$

$$j(z; q) = \sum_{k=0}^{n-1} (-1)^k q^{k(k-1)/2} z^k j((-1)^{n+1} q^{n(n-1)/2 + nk} z^n; q^{n^2}), \quad (2.22)$$

$$j(qx^3; q^3) + xj(q^2 x^3; q^3) = J_1 \frac{j(x^2; q)}{j(x; q)}, \quad (2.23)$$

$$j(x; q)j(y, q) = j(-xy; q^2)j(-qx^{-1}y; q^2) - xj(-xyq; q^2)j(-x^{-1}y; q^2), \quad (2.24)$$

$$\frac{j(y; q)}{j(-y; q)} - \frac{j(x; q)}{j(-x; q)} = 2x \frac{j(y/x; q^2)j(qxy; q^2)}{j(-x; q)j(-y; q)} \quad (2.25)$$

and

$$\frac{j(zx; q)}{j(x; q)} = \frac{J_n^3 j(z; q)}{J_1^3 j(x^n; q^n)} \sum_{k=0}^{n-1} x^k \frac{j(zx^n q^k; q^n)}{j(zq^k; q^n)}. \quad (2.26)$$

Next, we state results which can be found in [1, Eqs. (1.6) and (1.7)] or deduced from (1.7), (2.22) and (2.23).

**Proposition 2.7.** *If  $w$  is a primitive third root of unity and  $x \in \mathbb{C}^*$  is generic, then*

$$j(w; q) = (1 - w)J_3, \quad (2.27)$$

$$j(-w; q) = (1 + w) \frac{J_1^2 J_6}{J_2 J_3}, \quad (2.28)$$

$$j(-wq; q^2) = \frac{J_1 J_4 J_6^2}{J_2 J_3 J_{12}}, \quad (2.29)$$

$$j(-wq; q^3) = J_9 \left( \frac{j(q^2; q^9)}{j(-q; q^9)} - w^2 q \frac{j(q^8; q^9)}{j(-q^4; q^9)} \right), \quad (2.30)$$

$$j(-wq; q^6) = J_{18} \left( \frac{j(q^{10}; q^{18})}{j(-q^5; q^{18})} + wq \frac{j(q^{14}; q^{18})}{j(-q^7; q^{18})} \right) \quad (2.31)$$

and

$$j(x; q)j(xw; q)j(xw^2; q) = \frac{J_1^3}{J_3} j(x^3; q^3). \quad (2.32)$$

Before proceeding, recall that given a  $q$ -series  $F(q)$  with integral powers, its  $N$ -dissection is a representation of the form

$$F(q) = \sum_{k=0}^{N-1} q^k F_k(q^k)$$

where  $F_k(q)$  is a series in  $q$  with integral powers. The study of  $N$ -dissections ostensibly began in Ramanujan's lost notebook [23] and has now become its own extensive literature. As it is not possible to give a comprehensive overview of this area here, we only mention the classical result of Atkin and Swinnerton-Dyer on 5 and 7 dissections of the rank of partitions [2] and the recent impressive work [7] wherein the 11-dissection of the deviations of the rank and crank modulo 11 are obtained. Subsequent applications of [7] include new proofs of crank equalities [14], crank–crank inequalities [5, 13], congruences for  $p(n)$  modulo 11 [2] and linear rank congruences [3] as well as new results on rank and rank–crank inequalities, new congruences for rank and crank moments and for Andrews' smallest parts function. The final collection of results concerns 3-dissections of certain eta quotients. The following two identities are [4, Eq. (3.9)] and the  $a = -q$ ,  $b = -q^5$ ,  $n = 3$  case of [6, Entry 31, page 48], respectively.

**Proposition 2.8.** *We have*

$$\frac{1}{J_1^3} = W_0(q^3) + qW_1(q^3) + q^2W_2(q^3) \quad (2.33)$$

where

$$W_0(q) := \frac{J_3^9}{J_1^{12}} \left( \frac{1}{w^2(q)} + 8qw(q) + 16q^2w^4(q) \right),$$

$$W_1(q) := \frac{J_3^9}{J_1^{12}} \left( \frac{3}{w(q)} + 12qw^2(q) \right), \quad W_2(q) := 9 \frac{J_3^9}{J_1^{12}}$$

and

$$w(q) := \frac{J_1 J_6^3}{J_2 J_3^3}.$$

**Proposition 2.9.** *We have*

$$\frac{J_1 J_6}{J_2 J_3^2} = f_0(q^3) + qf_1(q^3) + q^2 f_2(q^3) \quad (2.34)$$

where

$$f_0(q) := \frac{j(q^7; q^{18})}{J_1 J_2}, \quad f_1(q) := -\frac{j(q^5; q^{18})}{J_1 J_2} \quad \text{and} \quad f_2(q) := -q \frac{j(q; q^{18})}{J_1 J_2}.$$

The following 3-dissections appear to be new. As the proofs are comparable, we only give details once.

**Proposition 2.10.** *We have*

$$\frac{J_2^4 J_8}{J_1 J_4^3} = g_0(q^3) + qg_1(q^3) + q^2 g_2(q^3) \quad (2.35)$$

where

$$g_0(q) := \frac{J_1 J_2^2 J_8^2 J_{12}^2}{J_4^5 J_{24}}, \quad g_1(q) := \frac{J_2^7 J_3 J_8^2 J_{12}^4}{J_1^2 J_4^7 J_6^3 J_{24}} \quad \text{and} \quad g_2(q) = -2 \frac{J_2^2 J_6^2 J_8^3}{J_3 J_4^5}.$$

*Proof.* If we multiply both sides of (2.35) by  $q^{12} \frac{J_1 J_4^3 J_6^6 J_{18}^8 J_{36}^2}{J_2^4 J_8}$ , then the claim is equivalent to

$$\begin{aligned} \eta^6(q^{12})\eta^8(q^{18})\eta^2(q^{36}) &= \frac{\eta(q)\eta(q^3)\eta^3(q^4)\eta^2(q^6)\eta(q^{12})\eta^8(q^{18})\eta^2(q^{24})\eta^4(q^{36})}{\eta^4(q^2)\eta(q^8)\eta(q^{72})} \\ &+ \frac{\eta(q)\eta^3(q^4)\eta^7(q^6)\eta(q^9)\eta^5(q^{18})\eta^2(q^{24})\eta^6(q^{36})}{\eta^4(q^2)\eta^2(q^3)\eta(q^8)\eta(q^{12})\eta(q^{72})} \\ &- 2 \frac{\eta(q)\eta^3(q^4)\eta^2(q^6)\eta(q^{12})\eta^{10}(q^{18})\eta^3(q^{24})\eta^2(q^{36})}{\eta^4(q^2)\eta(q^8)\eta(q^9)} \end{aligned} \quad (2.36)$$

where  $\eta(q) := q^{1/24} J_1$ . Using [9, Propositions 5.9.2 and 5.9.3], we see that both sides of (2.36) are holomorphic modular forms of level 72 and weight 8. Since the corresponding space of modular forms has Sturm bound 97 [9, Corollary 5.6.14], it suffices to check that the first 97 coefficients in the  $q$ -series expansion of (2.36) agree. This completes the proof of (2.35).  $\square$

**Proposition 2.11.** *We have*

$$\frac{J_2^3}{J_1 J_8} = h_0(q^3) + q h_1(q^3) + q^2 h_2(q^3) \quad (2.37)$$

where

$$h_0(q) := \frac{J_4^4 J_6^2}{J_2 J_3 J_8^3}, \quad h_1(q) := \frac{J_1 J_4 J_6 J_{24}}{J_8^2 J_{12}} \quad \text{and} \quad h_2(q) := -\frac{J_2^5 J_3 J_{12} J_{24}}{J_1^2 J_4 J_6^2 J_8^2}.$$

**Proposition 2.12.** *We have*

$$\frac{J_2}{J_4^2} = I_0(q^3) + q I_1(q^3) + q^2 I_2(q^3) \quad (2.38)$$

where

$$I_0(q) := \frac{J_2^2 J_6^3}{J_4^6}, \quad I_1(q) := q \frac{J_2^4 J_{12}^6}{J_4^8 J_6^3} \quad \text{and} \quad I_2(q) := -\frac{J_2^3 J_{12}^3}{J_4^7}.$$

### 3. PROOFS OF THEOREMS 1.1 AND 1.2

*Proof of Theorem 1.1.* We follow the strategy in [20]. Consider the left-hand side of (2.9) and take  $z' = -1$  in (2.10). This yields

$$\begin{aligned}
\frac{1}{M} \sum_{j=1}^{M-1} \zeta_M^{-aj} \overline{S}_d(\zeta_M^j; q) &= \frac{1}{M} \sum_{j=1}^{M-1} \zeta_M^{-aj} (1 - \zeta_M^j) (1 - 2m(\zeta_M^{-2j} q^{d^2}, q^{2d^2}, -1) + 2\Lambda(d, \zeta_M^j, z_0, -1)) \\
&= \frac{1}{M} \left( \sum_{j=0}^{M-1} \zeta_M^{-aj} - \sum_{j=0}^{M-1} \zeta_M^{-(a-1)j} \right) - \frac{2}{M} \left( \sum_{j=0}^{M-1} \zeta_M^{-aj} m(\zeta_M^{-2j} q^{d^2}, q^{2d^2}, -1) \right. \\
&\quad \left. - \sum_{j=0}^{M-1} \zeta_M^{-(a-1)j} m(\zeta_M^{-2j} q^{d^2}, q^{2d^2}, -1) \right) \\
&\quad + \frac{2}{M} \sum_{j=1}^M \zeta_{\frac{M}{2}}^{-\frac{aj}{2}} (1 - \zeta_M^j) \Lambda(d, \zeta_M^j, z_0, -1). \tag{3.1}
\end{aligned}$$

To prove (1.11), we first use (2.7) to observe that the first two sums in the second line of (3.1) equal 0 if  $a < M$  and 1 if  $a = M$ . For the third sum, we split it into two further sums. We then reindex the resulting second sum by  $j \rightarrow \frac{M}{2} + j$ , use that  $\zeta_M^{-2} = \zeta_{\frac{M}{2}}^{-1}$ , write  $a = 2t$  where  $t > 0$ , apply (2.3) and take  $k = t - 1$ ,  $n = \frac{M}{2}$ ,  $z = -1$ ,  $x = q^{-d^2}$  and  $q \rightarrow q^{2d^2}$  in (2.2) to obtain

$$\begin{aligned}
\sum_{j=0}^{M-1} \zeta_M^{-aj} m(\zeta_M^{-2j} q^{d^2}, q^{2d^2}, -1) &= 2 \sum_{j=0}^{\frac{M}{2}-1} \zeta_M^{-aj} m(\zeta_{\frac{M}{2}}^{-j} q^{d^2}, q^{2d^2}, -1) \\
&= M(-1)^{\frac{a}{2}-1} q^{-\frac{a^2}{4} + \frac{a}{2}(1-d^2)} m((-1)^{\frac{M}{2}+1} q^{\frac{M^2}{4} - \frac{aM}{2} + \frac{M}{2}(1-d^2)}, q^{\frac{M^2}{2}}, z') \\
&\quad + q^{-d^2} M \Psi_{\frac{a}{2}-1}^{\frac{M}{2}}(q^{-d^2}, -1, z'; q^2).
\end{aligned}$$

For the fourth sum, we similarly split it into two further sums, then reindex the resulting second sum by  $j \rightarrow \frac{M}{2} + j$ . We then use that  $\zeta_{\frac{M}{2}} = -1$  to obtain 0. In total, this yields (1.11).

To prove (1.12), we first use (2.7) to note that the first two sums in the second line of (3.1) equal 0. For the third sum, we first use that if  $\zeta_M$  is a primitive  $M$ -th root of unity, then so is  $\zeta_M^2$  as  $M$  is odd and then take  $k = \frac{2M-a}{2}$ ,  $n = M$ ,  $z = -1$ ,  $x = q^{d^2}$  and  $q \rightarrow q^{2d^2}$  in (2.2) to obtain

$$\begin{aligned}
\sum_{j=0}^{M-1} \zeta_M^{-aj} m(\zeta_M^{-2j} q^{d^2}, q^{2d^2}, -1) &= M q^{-d^2(\frac{2M-a}{2})^2} (-1)^{\frac{2M-a}{2}} m(q^{d^2 M(a-M)}, q^{2d^2 M^2}, z') \\
&\quad + M \Psi_{\frac{2M-a}{2}}^M(q^{d^2}, -1, z'; q^{2d^2}).
\end{aligned}$$

For the fourth sum, we take  $k = \frac{M+1-a}{2}$ ,  $n = M$ ,  $z = -1$ ,  $x = q^{d^2}$  and  $q \rightarrow q^{2d^2}$  in (2.2) to obtain

$$\begin{aligned}
\sum_{j=0}^{M-1} \zeta_M^{-(a-1)j} m(\zeta_M^{-2j} q^{d^2}, q^{2d^2}, -1) &= M q^{-d^2(\frac{a+M-1}{2})^2} (-1)^{\frac{a+M-1}{2}} m(q^{d^2(M-aM)}, q^{2d^2 M^2}, z'') \\
&\quad + M \Psi_{\frac{a+M-1}{2}}^M(q^{d^2}, -1, z''; q^{2d^2}).
\end{aligned}$$

In total, this yields (1.12).

Finally, to prove (1.13), we first use (2.7) to see that the first two sums in the second line of (3.1) equal 0 unless  $a = M$ , in which case the sum is  $M$ . For the third sum, we take  $k = \frac{M-a}{2}$ ,  $n = M$ ,  $z = -1$ ,  $x = q^{d^2}$  and  $q \rightarrow q^{2d^2}$  in (2.2) to obtain

$$\begin{aligned} \sum_{j=0}^{M-1} \zeta_M^{-aj} m(\zeta_M^{-2j} q^{d^2}, q^{2d^2}, -1) &= Mq^{-d^2(\frac{M-a}{2})^2} (-1)^{\frac{M-a}{2}} m(q^{d^2 a M}, q^{2d^2 M^2}, z') \\ &\quad + M\Psi_{\frac{M-a}{2}}^M(q^{d^2}, -1, z'; q^{2d^2}). \end{aligned}$$

For the fourth sum, we take  $k = \frac{2M-a+1}{2}$ ,  $n = M$ ,  $z = -1$ ,  $x = q^{d^2}$  and  $q \rightarrow q^{2d^2}$  in (2.2) to obtain

$$\begin{aligned} \sum_{j=0}^{M-1} \zeta_M^{-(a-1)j} m(\zeta_M^{-2j} q^{d^2}, q^{2d^2}, -1) &= Mq^{-d^2(\frac{2M-a+1}{2})^2} (-1)^{\frac{2M-a+1}{2}} m(q^{d^2 M(a-M-1)}, q^{2d^2 M^2}, z'') \\ &\quad + M\Psi_{\frac{2M-a+1}{2}}^M(q^2, -1, z''; q^{2d^2}). \end{aligned}$$

In total, this yields (1.13).  $\square$

*Proof of Theorem 1.2.* Consider the left-hand side of (2.9) and take  $z' = -1$  in (2.11). This yields

$$\begin{aligned} \frac{1}{M} \sum_{j=1}^{M-1} \zeta_M^{-aj} \overline{S}_d(\zeta_M^j; q) &= \frac{1}{M} \sum_{j=1}^{M-1} \zeta_M^{-aj} (1 - \zeta_M^j) \left( -1 + 2m((-1)^{\frac{d}{2}+1} \zeta_M^j q^{\frac{d^2}{4}}, q^{\frac{d^2}{2}}, -1) \right. \\ &\quad \left. + 2(-1)^{\frac{d}{2}} \zeta_M^j q^{-\frac{d^2}{4}} \Psi_0^{\frac{d}{2}}(\zeta_M^{\frac{2j}{d}} q^{1-d}, q, -1; q^2) \right) \\ &= -\frac{1}{M} \left( \sum_{j=0}^{M-1} \zeta_M^{-aj} - \sum_{j=0}^{M-1} \zeta_M^{-(a-1)j} \right) \\ &\quad + \frac{2}{M} \left( \sum_{j=0}^{M-1} \zeta_M^{-aj} m((-1)^{\frac{d}{2}+1} \zeta_M^j q^{\frac{d^2}{4}}, q^{\frac{d^2}{2}}, -1) \right. \\ &\quad \left. - \sum_{j=0}^{M-1} \zeta_M^{-(a-1)j} m((-1)^{\frac{d}{2}+1} \zeta_M^j q^{\frac{d^2}{4}}, q^{\frac{d^2}{2}}, -1) \right) \\ &\quad + \frac{2}{M} (-1)^{\frac{d}{2}} q^{-\frac{d^2}{4}} \sum_{j=1}^{M-1} \zeta_M^{j-aj} (1 - \zeta_M^j) \Psi_0^{\frac{d}{2}}(\zeta_M^{\frac{2j}{d}} q^{1-d}, q, -1; q^2). \end{aligned} \quad (3.2)$$

To prove (1.14), we first use (2.7) to observe that the first two sums in the second line of (3.2) equal 0 unless  $a = 1$ , in which case we obtain 1. For the third sum, we take  $k = a$ ,  $n = M$ ,

$z = -1$ ,  $x = (-1)^{\frac{d}{2}+1}q^{\frac{d^2}{4}}$  and  $q \rightarrow q^{\frac{d^2}{2}}$  in (2.2) to obtain

$$\begin{aligned} \sum_{j=0}^{M-1} \zeta_M^{-aj} m(\zeta_M^j (-1)^{\frac{d}{2}+1} q^{\frac{d^2}{4}}, q^{\frac{d^2}{2}}, -1) &= M(-1)^{\frac{da}{2}} q^{-\frac{d^2 a^2}{4}} m((-1)^{1+\frac{dM}{2}} q^{\frac{d^2}{4}(M^2-2Ma)}, q^{\frac{d^2 M^2}{2}}, z') \\ &+ M\Psi_a^M((-1)^{\frac{d}{2}+1} q^{\frac{d^2}{4}}, -1, z'; q^{\frac{d^2}{2}}). \end{aligned}$$

For the fourth sum, we take  $k = a - 1$ ,  $n = M$ ,  $z = -1$ ,  $x = (-1)^{\frac{d}{2}+1}q^{\frac{d^2}{4}}$  and  $q \rightarrow q^{\frac{d^2}{2}}$  in (2.2) to obtain

$$\begin{aligned} \sum_{j=0}^{M-1} \zeta_M^{-(a-1)j} m((-1)^{\frac{d}{2}+1} \zeta_M^j q^{\frac{d^2}{4}}, q^{\frac{d^2}{2}}, -1) \\ = M(-1)^{\frac{d}{2}(a-1)+1} q^{-\frac{d^2}{4}(a^2-2a+1)} m((-1)^{1+\frac{dM}{2}} q^{\frac{d^2}{4}(M^2-2M(a-1))}, q^{\frac{d^2 M^2}{2}}, z'') \\ + M\Psi_{a-1}^M((-1)^{\frac{d}{2}+1} q^{\frac{d^2}{4}}, -1, z''; q^{\frac{d^2}{2}}). \end{aligned}$$

In total, this yields (1.14).  $\square$

#### 4. SINGLE GENERALIZED RANK DEVIATIONS

In this section, we briefly discuss why there is no loss in generality in considering pairs of generalized rank deviations in Theorems 1.1 and 1.2. For  $M$  odd, we have

$$\overline{D}_d\left(\frac{M+1}{2}, M\right) + \overline{D}_d\left(\frac{M-1}{2}, M\right) = 2\overline{D}_d\left(\frac{M-1}{2}, M\right)$$

and so Theorem 1.1 can be used to find a formula for any single  $\overline{D}_d(a, M)$ . Precisely, we prove the following result.

**Proposition 4.1.** *Let  $M$  be odd. For  $0 \leq n \leq \frac{M+1}{2}$ , we have*

$$\begin{aligned} \overline{D}_d\left(\frac{M+1}{2} + n, M\right) &= \frac{1}{2} \left( \sum_{i=0}^n \left[ \overline{D}_d\left(\frac{M+1}{2} - n + 2i, M\right) + \overline{D}_d\left(\frac{M+1}{2} - n + 2i - 1, M\right) \right] \right. \\ &\quad \left. - \sum_{i=0}^{n-1} \left[ \overline{D}_d\left(\frac{M+1}{2} - n + 2i + 1, M\right) + \overline{D}_d\left(\frac{M+1}{2} - n + 2i, M\right) \right] \right). \end{aligned}$$

*Proof.* We begin by letting

$$\overline{D}_n := \sum_{i=0}^n \left[ \overline{D}_d\left(\frac{M+1}{2} + i, M\right) + \overline{D}_d\left(\frac{M-1}{2} - i, M\right) \right].$$

By (1.15), we have

$$\overline{D}_n = 2 \sum_{i=0}^n \overline{D}_d\left(\frac{M+1}{2} + i, M\right)$$

and so

$$\overline{D}_d\left(\frac{M+1}{2} + n, M\right) = \frac{\overline{D}_n - \overline{D}_{n-1}}{2}. \quad (4.1)$$

After reindexing, one obtains

$$\bar{D}_n = \sum_{i=0}^n \left[ \bar{D}_d \left( \frac{M+1}{2} - n + 2i, M \right) + \bar{D}_d \left( \frac{M+1}{2} - n + 2i - 1, M \right) \right]. \quad (4.2)$$

Combining (4.1) and (4.2) yields the result.  $\square$

For  $M$  even, one can use the following result.

**Proposition 4.2.** *Let  $M$  be even,  $0 \leq n \leq M-1$  and  $z_0, z' \in \mathbb{C}^*$  be generic. For  $d$  odd, we have*

$$\begin{aligned} \bar{D}_d(n, M) = \frac{(-1)^n}{M} \mathcal{O}_d(-1; q) + \frac{1}{M} \sum_{k=1}^{\frac{M}{2}-1} \left( \frac{1 - \zeta_M^k}{1 + \zeta_M^k} \right) (\zeta_M^{-kn} + \zeta_M^{kn}) \\ \times \left( 1 - 2m(\zeta_M^{-2k} q^{d^2}, q^{2d^2}, z') + 2\Lambda(d, \zeta_M^k, z_0, z') \right). \end{aligned} \quad (4.3)$$

For  $d$  even, we have

$$\begin{aligned} \bar{D}_d(n, M) = \frac{(-1)^n}{M} \mathcal{O}_d(-1; q) \\ + \frac{1}{M} \sum_{k=1}^{\frac{M}{2}-1} \left( \frac{1 - \zeta_M^k}{1 + \zeta_M^k} \right) (\zeta_M^{-kn} + \zeta_M^{kn}) \left( -1 + 2m((-1)^{\frac{d}{2}+1} \zeta_M^k q^{\frac{d^2}{4}}, q^{\frac{d^2}{2}}, z') \right. \\ \left. + 2(-1)^{\frac{d}{2}} \zeta_M^k q^{-\frac{d^2}{4}} \Psi_0^{\frac{d}{2}}(\zeta_M^{\frac{2k}{d}} q^{1-d}, q, z'; q^2) \right). \end{aligned} \quad (4.4)$$

*Proof.* From (1.3) and (2.7), one deduces

$$\bar{D}_d(n, M) = \frac{1}{M} \sum_{k=1}^{M-1} \mathcal{O}_d(\zeta_M^k; q) \zeta_M^{-kn} \quad (4.5)$$

for all  $M$ . For  $M$  even, we remove the  $k = \frac{M}{2}$  term from (4.5), split it into two further sums, then reindex the resulting second sum by  $k \rightarrow M - k$ . This yields

$$\bar{D}_d(n, M) = \frac{(-1)^n}{M} \mathcal{O}_d(-1; q) + \frac{1}{M} \sum_{k=1}^{\frac{M}{2}-1} (\zeta_M^{-kn} + \zeta_M^{kn}) \mathcal{O}_d(\zeta_M^k; q). \quad (4.6)$$

Finally, (4.3) and (4.4) follow from applying (2.8), (2.10) and (2.11) to (4.6).  $\square$

## 5. AN APPLICATION OF THEOREM 1.1

As an application of Theorem 1.1, we compute the 3-dissection of  $\mathcal{O}_3(\zeta_3; q)$  (cf. [17, Theorem 1.3]). In principle, one can reduce the number of eta quotients appearing in our result. We do not pursue this further. Let

$$\begin{aligned} A := j(-q^{12}; q^{27}), \quad B := qj(-q^{21}; q^{27}), \quad C := q^2j(-q^3; q^{27}), \quad D := \frac{j(q^{60}; q^{108})}{j(-q^{30}; q^{108})}, \\ E := q^6 \frac{j(q^{84}; q^{108})}{j(-q^{42}; q^{108})}, \quad F := \frac{j(q^{24}; q^{108})}{j(-q^{12}; q^{108})}, \quad G := q^{12} \frac{j(q^{96}; q^{108})}{j(-q^{48}; q^{108})}, \end{aligned} \quad (5.1)$$

$$\mathcal{G}_N := \sum_{\substack{k, l, m \in \{0,1,2\} \\ k+l+m \equiv N \pmod{3}}} q^{k+l+m} g_k(q^3) W_l(q^3) f_m(q^3) \quad (5.2)$$

and

$$\mathcal{H}_N := \sum_{\substack{k, l, m \in \{0,1,2\} \\ k+l+m \equiv N \pmod{3}}} q^{k+l+m} h_k(q^3) W_l(q^3) f_m(q^3) \quad (5.3)$$

where  $N = 0, 1$  or  $2$  and the functions  $W_i(q)$ ,  $f_i(q)$ ,  $g_i(q)$  and  $h_i(q)$  are given in (2.33)–(2.37). Also, recall the functions  $I_i(q)$  in (2.38).

**Theorem 5.1.** *We have*

$$\mathcal{O}_3(\zeta_3; q) = \bar{\mathcal{B}}_0(q^3) + q\mathcal{B}_1(q^3) + q^2\mathcal{B}_2(q^3) \quad (5.4)$$

where

$$\begin{aligned} \bar{\mathcal{B}}_0(q^3) &:= 6q^{-36} m(q^{-27}, q^{162}, -1) \\ &\quad - \frac{3}{2} q^{-9} \frac{J_{18} J_{27} J_{108} J_{162}^5}{J_{36}^2 J_{54} J_{81} J_{324}^3} \left( \frac{j(q^{27}; q^{162})}{j(-q^{27}; q^{162})} + \frac{j(q^{81}; q^{162})}{j(-q^{81}; q^{162})} \right) + \mathcal{B}_0(q^3) \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_N(q^3) &:= 3q^3 \frac{J_6^3 J_9 J_{108}}{J_3 J_{18} J_{36}} q^N I_N(q^3) \\ &\quad + \frac{J_3^2 J_6^2 J_{36}}{J_{12} J_{18}^2} \left( \frac{J_3^3 J_{12}^2 J_{18}^2 J_{72} J_{108}^2}{J_6^4 J_9 J_{24} J_{36} J_{54} J_{216}} \sum_{\substack{k, l \in \{0,1,2\} \\ k+l \equiv N \pmod{3}}} q^{k+l} g_k(q^3) W_l(q^3) \right. \\ &\quad - 2q^2 \frac{J_{12}^2 J_{108}}{J_6 J_{24}} \left( (2AD - AE) \mathcal{G}_{N+1} - (BD + BE) \mathcal{G}_N + (2CE - CD) \mathcal{G}_{2+N} \right) \\ &\quad + q^5 \frac{J_3^3 J_{18} J_{24} J_{36}^2 J_{216}}{J_6^3 J_9 J_{12} J_{72} J_{108}} \sum_{\substack{k, l \in \{0,1,2\} \\ k+l \equiv N+1 \pmod{3}}} q^{k+l} h_k(q^3) W_l(q^3) \\ &\quad \left. - 2q \frac{J_{24} J_{108}}{J_{12}} \left( (2AG + AF) \mathcal{H}_{N+2} - (2BF + BG) \mathcal{H}_{N+1} - (CG - CF) \mathcal{H}_N \right) \right) \end{aligned}$$

for  $N = 0, 1$  and  $2$ .

*Proof of Theorem 5.1.* We begin by decomposing  $\mathcal{O}_3(\zeta_3; q)$  as follows. By (1.3)–(1.5) and (1.15), we have

$$\begin{aligned} \mathcal{O}_3(\zeta_3; q) &= \sum_{n \geq 0} \sum_{s=0}^2 \bar{N}_3(s, 3, n) \zeta_3^s q^n \\ &= \sum_{n \geq 0} (\bar{N}_3(0, 3, n) + \zeta_3 \bar{N}_3(1, 3, n) + \zeta_3^2 \bar{N}_3(2, 3, n)) q^n \\ &= \sum_{n \geq 0} (\bar{N}_3(0, 3, n) - \bar{N}_3(2, 3, n)) q^n \\ &= \bar{D}_3(3, 3) + \bar{D}_3(2, 3) - (\bar{D}_3(2, 3) + \bar{D}_3(1, 3)) \end{aligned} \quad (5.5)$$

where we have used that  $\overline{N}_3(1, 3, n) = \overline{N}_3(2, 3, n)$  and  $1 + \zeta_3 + \zeta_3^2 = 0$ . We now take  $d = 3$ ,  $a = M = 3$ ,  $z' = z'' = z_0 = -1$  in (iii) and  $d = 3$ ,  $a = 2$ ,  $M = 3$ ,  $z' = z'' = z_0 = -1$  in (ii) of Theorem 1.1, respectively, and simplify using (2.3), (2.5) and the fact that

$$\Psi_0^3(q^9, -1, -1; q^{18}) = 0$$

to obtain from (5.5)

$$\begin{aligned} \mathcal{O}_3(\zeta_3; q) &= 6q^{-36}m(q^{-27}, q^{162}, -1) - 2\zeta_3\Lambda(3, \zeta_3, -1, -1) - 2\zeta_3^2\Lambda(3, \zeta_3^2, -1, -1) \\ &\quad + 4\Psi_2^3(q^9, -1, -1; q^{18}) - 2\Psi_1^3(q^9, -1, -1; q^{18}). \end{aligned} \quad (5.6)$$

By (1.10), (1.17), (1.18) and (2.20), we have

$$\begin{aligned} -2\zeta_3\Lambda(3, \zeta_3, -1, -1) &= -2q^{-1}\zeta_3^{5/3}\Psi_1^3(\zeta_3^{-2/3}q^3, -1, -1; q^2) \\ &\quad + \frac{2\zeta_3^{5/3}J_2^3}{3qj(-1; q^2)} \sum_{t=0}^2 \zeta_3^{-t} \frac{j(-\zeta_3^{t+1/3}/q; q^2)j(-\zeta_3^{-t-1/3}q^2; q^2)}{j(\zeta_3^{t+1/3}/q; q^2)j(\zeta_3^{-t-1/3}q^2; q^2)j(-\zeta_3^{-2t-2/3}q^3; q^2)} \\ &= -\frac{\zeta_3 J_2 J_6 J_{18}^4}{2J_4^2 J_{36}^2 j(-\zeta_3 q^9; q^{18})} \left( \frac{j(\zeta_3 q^{15}; q^{18})}{j(-\zeta_3 q^{15}; q^{18})} + \frac{j(\zeta_3 q^{21}; q^{18})}{j(-\zeta_3 q^{21}; q^{18})} \right) \\ &\quad + \frac{\zeta_3 J_2^4}{3J_4^2} \sum_{t=0}^2 \frac{j(-\zeta_3^{t+1/3}q; q)}{j(\zeta_3^{t+1/3}q; q)j(-\zeta_3^{2t+2/3}q; q^2)}. \end{aligned} \quad (5.7)$$

Similarly,

$$\begin{aligned} -2\zeta_3^2\Lambda(3, \zeta_3^2, -1, -1) &= \frac{\zeta_3^2 J_2 J_6 J_{18}^4}{2J_4^2 J_{36}^2 j(-\zeta_3^2 q^9; q^{18})} \left( \frac{j(\zeta_3 q^{15}; q^{18})}{j(-\zeta_3 q^{15}; q^{18})} + \frac{j(\zeta_3 q^{21}; q^{18})}{j(-\zeta_3 q^{21}; q^{18})} \right) \\ &\quad - \frac{\zeta_3^2 J_2^4}{3J_4^2} \sum_{t=0}^2 \frac{j(-\zeta_3^{t+2/3}q; q)}{j(\zeta_3^{t+2/3}q; q)j(-\zeta_3^{2t+4/3}q; q^2)} \end{aligned} \quad (5.8)$$

after applying (1.17), (1.18) and (2.21). Using (1.17), (1.18), the  $x = \zeta_3^2 q^{15}$  and  $y = \zeta_3 q^{15}$  case of (2.25), (2.28) and (2.32), one confirms

$$\frac{j(\zeta_3 q^{15}; q^{18})}{j(-\zeta_3 q^{15}; q^{18})} + \frac{j(\zeta_3 q^{21}; q^{18})}{j(-\zeta_3 q^{21}; q^{18})} = -2q^3 \zeta_3^2 (1 - \zeta_3^2) \frac{J_6^2 J_9^2 J_{36}^2 J_{54}^2}{J_3 J_{18}^6 J_{27}}. \quad (5.9)$$

Hence, taking the sum of the first terms on the right-hand sides of (5.7) and (5.8), respectively, combined with (2.29) and (5.9) yields after simplification

$$-(\zeta_3 - \zeta_3^2) \frac{J_2 J_6 J_{18}^4}{2J_4^2 J_{36}^2 j(-\zeta_3 q^9; q^{18})} \left( \frac{j(\zeta_3 q^{15}; q^{18})}{j(-\zeta_3 q^{15}; q^{18})} + \frac{j(\zeta_3 q^{21}; q^{18})}{j(-\zeta_3 q^{21}; q^{18})} \right) = 3q^3 \frac{J_2 J_6^3 J_9 J_{108}}{J_3 J_4^2 J_{18} J_{36}}. \quad (5.10)$$

It now remains to consider the sum of the second terms on the right-hand sides of (5.7) and (5.8), respectively. This is

$$(\zeta_3 - \zeta_3^2) \frac{J_2^4}{3J_4^2} \sum_{t=0}^2 \frac{j(-\zeta_3^{t+1/3}q; q)}{j(\zeta_3^{t+1/3}q; q)j(-\zeta_3^{2t+2/3}q; q^2)}. \quad (5.11)$$

We now apply (1.18) to the summand of (5.11), then take  $x = \zeta_3^{t+1/3}$ ,  $z = -1$  and  $n = 3$  in (2.26) to obtain

$$-(\zeta_3 - \zeta_3^2) \frac{2J_2^6 J_3^3}{3J_1^4 J_4^2 j(\zeta_3; q^3)} \sum_{k=0}^2 \frac{j(-\zeta_3 q^k; q^3)}{j(-q^k; q^3)} \sum_{t=0}^2 \frac{\zeta_3^{(t+1/3)k}}{j(-\zeta_3^{2(t+1/3)} q; q^2)}. \quad (5.12)$$

By (2.32),

$$j(-\zeta_3^{2/3} q; q^2) j(-\zeta_3^{8/3} q; q^2) j(-\zeta_3^{14/3} q; q^2) = \frac{J_2^3 J_3 J_{12} J_{18}^2}{J_6^2 J_9 J_{36}} \quad (5.13)$$

and so using (2.27) and (5.13) turns (5.12) into

$$-\zeta_3 \frac{2J_2^3 J_3^2 J_6^2 J_{36}}{3J_1^4 J_4^2 J_{12} J_{18}^2} \sum_{k=0}^2 \frac{j(-\zeta_3 q^k; q^3)}{j(-q^k; q^3)} \sum_{t=0}^2 \zeta_3^{(t+1/3)k} j(-\zeta_3^{2(t+4/3)} q; q^2) j(-\zeta_3^{2(t+7/3)} q; q^2). \quad (5.14)$$

Note that

$$\begin{aligned} & j(-\zeta_3^{2t+8/3} q; q^2) j(-\zeta_3^{2t+5/3} q; q^2) \\ &= \frac{J_2 J_8 J_{12}^2}{J_4 J_6 J_{24}} j(-\zeta_3^{4(t+1/3)} q^2; q^4) - \zeta_3^{2t+2/3} q \frac{J_4^2 J_{24}}{J_8 J_{12}} j(-\zeta_3^{4(t+1/3)} q^4; q^4) \end{aligned}$$

by (2.24), (2.27) and (2.29) and so (5.14) equals

$$\begin{aligned} & -\frac{2J_2^3 J_3^2 J_6^2 J_{36}}{3J_1^4 J_4^2 J_{12} J_{18}^2} \left( \zeta_3 \frac{J_2 J_8 J_{12}^2}{J_4 J_6 J_{24}} \sum_{k=0}^2 \frac{j(-\zeta_3 q^k; q^3)}{j(-q^k; q^3)} \sum_{t=0}^2 \zeta_3^{(t+1/3)k} j(-\zeta_3^{4(t+1/3)} q^2; q^4) \right. \\ & \left. - q \frac{J_4^2 J_{24}}{J_8 J_{12}} \sum_{k=0}^2 \zeta_3^{(k+5)/3} \frac{j(-\zeta_3 q^k; q^3)}{j(-q^k; q^3)} \sum_{t=0}^2 \zeta_3^{(k+2)t} j(-\zeta_3^{(t+4/3)} q^4; q^4) \right). \end{aligned} \quad (5.15)$$

We now simplify the first line in (5.15). By (1.7) and (2.7),

$$\begin{aligned} \sum_{t=0}^2 \zeta_3^{(t+1/3)k} j(-\zeta_3^{4(t+1/3)} q^2; q^4) &= \sum_{n \in \mathbb{Z}} q^{2n^2} \sum_{t=0}^2 \zeta_3^{(t+1/3)(4n+k)} \\ &= 3 \sum_{n \equiv 2k \pmod{3}} q^{2n^2} \zeta_3^{(4n+k)/3} \\ &= 3 \sum_{s \in \mathbb{Z}} q^{36s(s-1)/2 + (24k+18)s + 8k^2} \zeta_3^s \\ &= 3q^{8k^2} j(-\zeta_3 q^{24k+18}; q^{36}). \end{aligned} \quad (5.16)$$

First, applying (2.29) to (5.16), the  $k = 0$  term in the first line of (5.15) equals

$$-\frac{3J_3^3 J_{18}^2 J_{72} J_{108}^2}{2J_6^3 J_9 J_{36} J_{54} J_{216}}. \quad (5.17)$$

Next, applying (2.31) to (5.16) and the  $z = -\zeta_3 q$ ,  $q \rightarrow q^3$  and  $n = 3$  case of (2.22) and simplifying, the  $k = 1$  term in the first line of (5.15) equals

$$3q^2 \frac{J_1 J_6 J_{108}}{J_2 J_3^2} (AD - BE - CD + CE + \zeta_3(AE + BD - BE - CD)) \quad (5.18)$$

where  $A, B, C, D$  and  $E$  are given in (5.1). Finally, applying (1.17), (1.18) and (2.31) to (5.16) and simplifying, the  $k = 2$  term in the first line of (5.15) equals

$$3q^2 \frac{J_1 J_6 J_{108}}{J_2 J_3^2} (AD - AE - BD + CE + \zeta_3(-AE - BD + BE - CD)). \quad (5.19)$$

Combining (5.17)–(5.19) and simplifying, the first line of (5.15) equals

$$-\frac{J_2^4 J_3^2 J_6 J_8 J_{12} J_{36}}{J_1^4 J_4^3 J_{18}^2 J_{24}} \left( -\frac{J_3^3 J_{18}^2 J_{72} J_{108}^2}{J_6^3 J_9 J_{36} J_{54} J_{216}} + 2q^2 \frac{J_1 J_6 J_{108}}{J_2 J_3^2} (2AD - AE - BD - BE - CD + 2CE) \right). \quad (5.20)$$

We now simplify the second line in (5.15). Similar to (5.16), we find

$$\sum_{t=0}^2 \zeta_3^{(k+2)t} j(-\zeta_3^{t+4/3}; q^4) = 3\zeta_3^{(5k-5)/3} q^{2k^2-6k+4} j(-\zeta_3 q^{36-12k}; q^{36}). \quad (5.21)$$

Applying (2.28) to (5.21), the  $k = 0$  term in the second line of (5.15) equals

$$3q^4 \frac{J_3^3 J_{18} J_{36}^2 J_{216}}{2J_6^3 J_9 J_{72} J_{108}}. \quad (5.22)$$

Next, applying (1.18) and (2.30) to (5.21) and (again) the  $z = -\zeta_3 q$ ,  $q \rightarrow q^3$  and  $n = 3$  case of (2.22) and simplifying, the  $k = 1$  term in the second line of (5.15) becomes

$$3 \frac{J_1 J_6 J_{108}}{J_2 J_3^2} (-AF - AG + BF + CG + \zeta_3(-AF - BG + CF + CG)) \quad (5.23)$$

where  $F$  and  $G$  are given in (5.1). Finally, one can similarly show that the  $k = 2$  term in the second line of (5.15) is

$$3 \frac{J_1 J_6 J_{108}}{J_2 J_3^2} (-AG + BF + BG - CF + \zeta_3(AF + BG - CF - CG)). \quad (5.24)$$

Combining (5.22)–(5.24) and simplifying, the second line of (5.15) equals

$$-q \frac{J_2^3 J_3^2 J_6^2 J_{24} J_{36}}{J_1^2 J_4^2 J_8 J_{12}^2 J_{18}^2} \left( q^4 \frac{J_3^3 J_{18} J_{36}^2 J_{216}}{J_6^3 J_9 J_{72} J_{108}} + 2 \frac{J_1 J_6 J_{108}}{J_2 J_3^2} (-2AG + 2BF + BG - AF - CF + CG) \right). \quad (5.25)$$

As (5.15) is the sum of (5.20) and (5.25), we obtain

$$\begin{aligned} & \frac{J_2^3 J_3^2 J_6^2 J_{36}}{J_1^4 J_4^2 J_{12} J_{18}^2} \left( \frac{J_2 J_8 J_{12}^2}{J_4 J_6 J_{24}} \left( \frac{J_3^3 J_{18}^2 J_{72} J_{108}^2}{J_6^3 J_9 J_{36} J_{54} J_{216}} - 2q^2 \frac{J_1 J_6 J_{108}}{J_2 J_3^2} (2AD - AE - BD - BE - CD + 2CE) \right) \right. \\ & \left. + q \frac{J_4^2 J_{24}}{J_8 J_{12}} \left( q^4 \frac{J_3^3 J_{18} J_{36}^2 J_{216}}{J_6^3 J_9 J_{72} J_{108}} + 2 \frac{J_1 J_6 J_{108}}{J_2 J_3^2} (-2AG + 2BF + BG - AF - CF + CG) \right) \right). \end{aligned} \quad (5.26)$$

Note that (5.26) can be rearranged as

$$\begin{aligned} & \frac{J_3^2 J_6^2 J_{36}}{J_{12} J_{18}^2} \left( \frac{J_3^3 J_{12}^2 J_{18}^2 J_{72} J_{108}^2}{J_6^4 J_9 J_{24} J_{36} J_{54} J_{216}} \frac{J_2^4 J_8}{J_1 J_4^3 J_1^3} \frac{1}{J_1^3} \right. \\ & \quad - 2q^2 \frac{J_{12}^2 J_{108}}{J_6 J_{24}} (2AD - AE - BD - BE - CD + 2CE) \frac{J_2^4 J_8}{J_1 J_4^3 J_1^3} \frac{1}{J_2 J_3^2} \frac{J_1 J_6}{J_2 J_3^2} \\ & \quad + q^5 \frac{J_3^3 J_{18} J_{24} J_{36}^2 J_{216}}{J_6^3 J_9 J_{12} J_{72} J_{108}} \frac{J_2^3}{J_1 J_8} \frac{1}{J_1^3} \\ & \quad \left. + 2q \frac{J_{24} J_{108}}{J_{12}} (-2AG + 2BF + BG - AF - CF + CG) \frac{J_2^3}{J_1 J_8} \frac{1}{J_1^3} \frac{J_1 J_6}{J_2 J_3^2} \right). \end{aligned} \tag{5.27}$$

Moreover, using (1.9), (1.17), (1.18) and simplifying, one can check

$$\begin{aligned} & 4\Psi_2^3(q^9, -1, -1; q^{18}) - 2\Psi_1^3(q^9, -1, -1; q^{18}) \\ & \quad = -\frac{3}{2} q^{-9} \frac{J_{18} J_{27} J_{108} J_{162}^5}{J_{36}^2 J_{54} J_{81} J_{324}^3} \left( \frac{j(q^{27}; q^{162})}{j(-q^{27}; q^{162})} + \frac{j(q^{81}; q^{162})}{j(-q^{81}; q^{162})} \right). \end{aligned} \tag{5.28}$$

We now insert (2.33)–(2.37) into (5.27) and (2.38) into (5.10) and combine with (5.28). After substituting the resulting expressions into (5.6) and recalling (5.2) and (5.3), we arrive at (5.4). This proves the result.  $\square$

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